Assessed Problems Collection 1

Please give all your answers from week 2 as a single notebook. Answers to other questions should be submitted in a single file preferably written latex but a handwritten file is also accepted as long as it is clearly readable and clean enough. If you used Mathematica for any other problems, please include it in your notebook. (Do not forget to check that your code actually runs before submitting!).

You can work in group of 4 people (at most). If you work in group, you must give the name of your colleagues at the beginning of your manuscript (only one member can submit your answers for the group).

Submission deadline: 15th April 2024 at 9:30 AM

Late submissions are penalised by -10% of the total number of points and -20% for each additional day (e.g. if you submit on the 16th April 2024 at 10:00 AM you will lose 30% of the total number of points).

1. Week 2 mathematica notebook.

2. (Canonical purification) Let ρ_A be a density operator and let $\sqrt{\rho_A}$ be its unique positive semi-definite square root (i.e., $\rho_A = \sqrt{\rho_A}\sqrt{\rho_A}$.) We define the canonical purification of ρ_A as follows:

$$|\psi^\rho\rangle_{RA} := (I_R \otimes \sqrt{\rho_A})|\Omega\rangle_{RA}\,,$$

where $\dim(R) = \dim(A)$ and $|\Omega\rangle_{RA} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_A$ is the unnormalized maximally entangled vector. Show that $|\psi^\rho\rangle_{RA}$ is a purification of ρ_A.

3. In this problem we will learn the concept of convexity and extreme points of operators. We will show that pure states are extreme points of the convex set of states and orthogonal measurements are extreme points of the convex set of 2-outcome POVMs. We start with the definition of extreme points.

Consider the space of bounded operators $\mathcal{B}$, for any operators A, O_1, O_2, we say that A ***lies between*** O_1 and O_2 if $O_1 \neq O_2$ and there exists a $0 < p < 1$ such that $A = pO_1 + (1-p)O2$. If $\mathcal{H}$ is subspace of $\mathcal{B}$ and $A \in \mathcal{H}$, we call A is ***an extreme point*** of $\mathcal{H}$ if it does not lie between any two distinct points of $\mathcal{H}$. That is $A = pO_1 + (1-p)O2$ if and only if $A = O_1$ ($p = 1$) or $A = O_2$ ($p = 0$).

a) Show that extreme points of the set of quantum states are pure states, and pures states are extreme points.

Let $F = \{F_1, F_2\}$ and $G = \{G_1, G_2\}$ be two POVMs. We define an element-wise convex combination of F and G as $\alpha F + (1-\alpha)G := \{\alpha F_1 + (1-\alpha)G_1, \alpha F_2 + (1-\alpha)G_2\}$, with $0 \leqslant \alpha \leqslant 1$.

b) Consider a POVM with two outcomes and respective measurement operators E and $\mathbb{1} - E$. Suppose that E has an eigenvalue λ such that $0 < \lambda < 1$. Show that the POVM is not extremal by expressing it as a nontrivial convex combination of two POVMs.
Hint: Consider the spectral decomposition of E and rewrite it as a convex combination of two POVM elements.

c) Suppose that E is an orthogonal projector. Show that the POVM cannot be expressed as a nontrivial convex combination of POVMs.

d) What is the operational interpretation of an element-wise convex combination of POVMs?

4. (Short answer question from Exam 2024)

Consider a measure-and-update channel $\mathcal{E}(\rho) = \sum_k |k\rangle\langle k| \otimes B_k \rho B_k^\dagger$ where ρ is the initial state to be measured, $\{|k\rangle\}$ is an orthonormal basis and $\{B_k\}$ are a set of Kraus operators.

   (a) Write down the Kraus operators for the channel $\mathcal{E}$. Explain why this is called the 'Measure-and-update' channel.

Now consider the following choice in operators $\{B_k\}$:

$$B_1 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where λ is real and $0 \leqslant \lambda \leqslant 1$.

Further suppose the initial state is the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(b) What is the probability of measuring outcomes 1 and 2? What are the corresponding output states of the system that is measured? (5 marks)

Suppose you now instead have the operators $\{B'_k\}$:

$$B'_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}, \quad B'_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where λ is defined above.

(c) What are the probabilities and output states in this case?

(d) What do you conclude from comparing your answers in (b) and (c)?

5. **Normal form of single qubit channel.** In general, any single-qubit channel $\mathcal{N}$ can be written in the following *normal form*

$$\mathcal{N}(\cdot) = U\mathcal{N}'(V(\cdot)V^\dagger)U^\dagger, \tag{3}$$

where U, V are arbitrary single-qubit unitaries. The action of $\mathcal{N}'$ on the single-qubit Pauli matrices is given by

$$\mathcal{N}'(I) = I + t_X X + t_Y Y + t_Z Z, \tag{4}$$
$$\mathcal{N}'(X) = D_X X, \tag{5}$$
$$\mathcal{N}'(Y) = D_Y Y, \tag{6}$$
$$\mathcal{N}'(Z) = D_Z Z, \tag{7}$$

where $\mathbf{D} = (D_X, D_Y, D_Z) \in [-1,1]^3$ and $\mathbf{t} = (t_X, t_Y, t_Z) \in [-1,1]^3$ are two vectors, which we refer as *normal form parameters*. We say that $\mathcal{N}$ is *unital* if $\mathcal{N}(I) = I$ and *non-unital* otherwise.

For simplicity here we assume $U = I = V$, and hence $\mathcal{N}(\cdot) = \mathcal{N}'(\cdot)$.

(a) Write down the general expression of the Choi state for the channel $\mathcal{N}$.

(b) If $\mathcal{N}$ is a depolarizing channel, then it can be written as $\mathcal{N}(\rho) = p\mathrm{Tr}[\rho]\frac{I}{2} + (1-p)\rho$ with $0 \leqslant p \leqslant 1$. Is this channel unital or non-unital?

(c) Decompose ρ using Bloch vector representation and explain the effect of the depolarizing channel in the Bloch sphere.

(d) Find the conditions on $\mathbf{D}$ and $\boldsymbol{t}$, such that $\mathcal{N}$ is a depolarizing channel.

(e) Find the Kraus operators for this channel?
Hint: Of course you can use Choi state to do this. But maybe it's easier to use the equality $\mathcal{I} = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z)$ you have proved in problem sheet 2.

(f) Consider a channel such that $\mathbf{D} = (\sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma)$ and $\boldsymbol{t} = (0,0,\gamma)$ where $0 < \gamma \leqslant 1$. Is this channel unital or non-unital? What is the effect of this channel on a generic single qubit state? What happens if this channel is applied $N \to \infty$ times in succession?

(g) Find the Kraus operators corresponding to the channel with $\mathbf{D} = (\sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma)$ and $\boldsymbol{t} = (0,0,\gamma)$ where $0 < \gamma \leqslant 1$.

(h) Now consider $D = (a,a,b)$ and $t = (0,0,c)$. Find the Kraus operators in this case. Show that they reduce to the standard expressions for both depolarizing and amplitude damping channel Kraus operators for appropriate values of a, b, c.
Hint: Mathematica might be helpful but not mandatory.

6. Let us first consider a circuit constructed by consecutive layers of the following n-qubits circuit

$$U(\boldsymbol{\theta}^{(k)}) = \left(\prod_{i=1}^{n-1} CZ_{i,i+1} \otimes \mathbb{1}^{n-2}\right)\left(\bigotimes_{i=1}^{n} R_{X_i}(\theta^{(k)}_{2n+i})\right)\left(\bigotimes_{i=1}^{n} R_{Y_i}(\theta^{(k)}_{n+i})\right)\left(\bigotimes_{i=1}^{n} R_{X_i}(\theta^{(k)}_{i})\right) , \tag{8}$$

where we recall $R_\sigma(\theta) = \exp\left(-i\frac{\theta}{2}\sigma\right)$ and X_i, Y_i are respectively the Pauli operators X and Y applied on the i-th qubit. Moreover, $CZ_{i,j}$ is a controlled-Z gate (2-qubits gate) applied on qubits i and j.

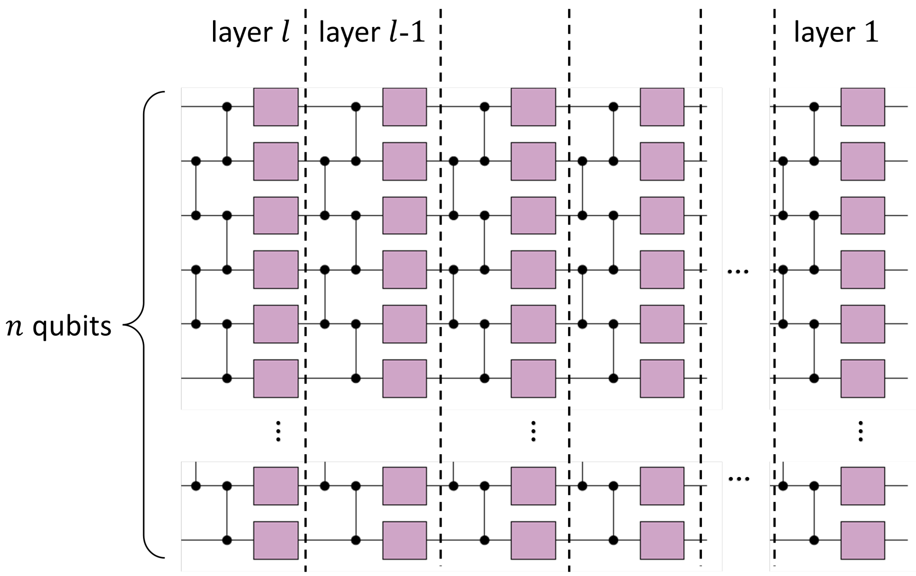

FIG. 1. **A parametrized circuit.** This plot illustrates the parametrized circuit considered in this problem. The circuit is drawn in Heiseberg picture (input state on the right and observable on the left). The pink boxes represent the collective single qubit rotation within each layer $R_{X_i}(\theta^{(k)}_{2n+i})R_{Y_i}(\theta^{(k)}_{n+i})R_{X_i}(\theta^{(k)}_{i})$, which followed by Controlled-Z gates on neighbouring qubits.

Now, as illustrated in Fig. 1, the total circuit $V = V(\boldsymbol{\theta})$ with $\boldsymbol{\theta} := (\boldsymbol{\theta}^{(1)}, \cdots \boldsymbol{\theta}^{(l)})$ is an l-layer circuit where each layer corresponds to $U\left(\boldsymbol{\theta}^{(k)}\right)$ with $\boldsymbol{\theta}^{(k)}$ the set of $3n$ parameters for layer k (for simplicity we assume all $3nl$ parameters are uncorrelated).

$$V = U\left(\boldsymbol{\theta}^{(l)}\right)\cdots U\left(\boldsymbol{\theta}^{(2)}\right)U\left(\boldsymbol{\theta}^{(1)}\right) . \tag{9}$$

We are interested in computing the average (over the parameters $\boldsymbol{\theta}$) of the expectated value of the Pauli $Z \otimes \mathbb{1}^{\otimes n-1}$ after applying the circuit V to the zero state i.e. we want to compute

$$\mu = \mathbb{E}_{\boldsymbol{\theta}}[\langle \mathbf{0}|V^\dagger(Z \otimes \mathbb{1}^{\otimes n-1})V|\mathbf{0}\rangle] , \tag{10}$$

Compute this quantity for an arbitrary number of layers l where each parameter is distributed i.i.d according to

(a) a Gaussian distribution with zero mean and variance Δ.

(b) a sine distribution over 0 and π (i.e. $\Pr(\theta) = \frac{\sin(\theta)}{2}$ for $0 \leqslant \theta \leqslant \pi$ and $\Pr(\theta) = 0$ otherwise).

Hint: use vectorization and the Pauli transfer matrix formalism.