Quantum Information and Quantum Computing, Solutions 1

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Problem 1 : Matrix-vector and matrix-matrix products

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}=\begin{pmatrix}1/2&0&0&0\\0&1/2&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&1\\0&1&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&1\\0&1&1&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&1&2\\0&1&2&1\\0&2&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\end{pmatrix}\times\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}=\begin{pmatrix}1&1&1&1\end{pmatrix}$$

Problem 2 : Eigenvalues and eigenvectors

In this problem we are requested to find the set of eigenvalues λ_i with corresponding eigenvectors $\mathbf{v}_i$ of a matrix M such that

$$M\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

We start by finding the roots of the characteristic polynomial $\det(M-\lambda I)$.

For the matrix $M = \begin{pmatrix}1 & i\\ 2 & 1\end{pmatrix}$ it is given by

$$\det(M-\lambda I) = \det(\begin{pmatrix}1\text{-}\lambda & i\\ 2 & 1\text{-}\lambda\end{pmatrix}) = (1-\lambda)^2 - 2i \quad .$$

Setting it to 0 and expanding $2i = (i+1)^2$ we arrive at the solutions $\lambda_{1,2} = 1 \pm (i+1)$ which constitute the eigenvalues of M.

Next we can find the eigenvectors $\mathbf{v}_i$ corresponding to each eigenvalue λ_i by solving the linear system of equations $(M-\lambda_i I)\mathbf{v}_i = \mathbf{0}$.

For the first eigenvalue $\lambda_1 = 2 + i$ we have that

$$M - \lambda_1 I = \begin{pmatrix} 1\text{-}(2{+}i) & i \\ 2 & 1\text{-}(2{+}i) \end{pmatrix} = \begin{pmatrix} \text{-}1\text{-}i & i \\ 2 & \text{-}1\text{-}i \end{pmatrix}$$

this gives us the redundant system of equations

$$\begin{pmatrix} \text{-}1\text{-}i & i \\ 2 & \text{-}1\text{-}i \end{pmatrix} \mathbf{v}_1 = \mathbf{0}$$

with a particular solution given by the normalized eigenvector $\mathbf{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1{+}i \\ 2 \end{pmatrix}$.

The complete set of solutions (eigenspace) is given by the $\text{span}(\mathbf{v}_1) = \left\{ c \cdot \mathbf{v}_1 \mid c \in \mathbb{C} \setminus \{0\} \right\}$

Similarly for the second eigenvalue $\lambda_2 = -i$ we find the normalized eigenvector $\mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} \text{-}1\text{-}i \\ 2 \end{pmatrix}$

Finally with the matrix of eigenvectors

$$U = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1{+}i & \text{-}1\text{-}i \\ 2 & 2 \end{pmatrix}$$

and the diagonal matrix of eigenvalues

$$\Lambda = \text{diag}(\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2{+}i & 0 \\ 0 & \text{-}i \end{pmatrix}$$

we can easily verify that M can be diagonalized as

$$M = U \Lambda U^{-1} \tag{1}$$

Problem 3 : Pauli matrices

To solve this problem we can apply the definition of product between matrices and verify the equivalences with the results of calculations.

Problem 4 : Exponential of a matrix

From the definition of the matrix-matrix product it is easy to see that powers of a diagonal matrix $\Lambda_{ij} = \delta_{ij} \Lambda_{ii}$ can be calculated by taking the element-wise power of the diagonal:

$$(\Lambda^2)_{ij} = \sum_k \Lambda_{ik} \Lambda_{kj} = \sum_k \delta_{ik} \Lambda_{ii} \delta_{kj} \Lambda_{jj} = \delta_{ij} \Lambda_{ii}^2$$

Consequently, with the matrix exponential being defined as the sum of powers

$$e^M \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

we have that the matrix exponential of a diagonal matrix amounts to taking the element-wise exponential of the diagonal.

We notice that the Pauli matrix $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal and so it's matrix exponential is given by

$$e^{i\alpha Z} = \exp \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Next we want to calculate the matrix exponential of $i\alpha Y$ which is given by

$$e^{i\alpha Y} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\alpha)^n Y^n$$

In the previous problem we saw that $Y^2 = I$, thus we have that for even powers $Y^{2n} = I$ and for odd powers $Y^{2n+1} = Y$. We can split the sum into it's even and odd parts:

$$\sum_{n=0}^{\infty} \frac{1}{n!}(i\alpha)^n Y^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(i\alpha)^{2n} I + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(i\alpha)^{2n+1} Y$$

we have that $(i\alpha)^{2n} = (-1)^n \alpha^{2n}$ and $(i\alpha)^{2n+1} = i(-1)^n \alpha^{2n+1}$ and so we can decompose it further(while also taking out the now constant matrices I and Y and the factor i)

$$= I \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n (\alpha)^{2n}}_{=\cos(\alpha)} + iY \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^n (\alpha)^{2n+1}}_{=\sin(\alpha)}$$

Seeing that the remaining power series are those of sin and cos respectively we get that the matrix exponential of $i\alpha Y$ is given by

$$e^{i\alpha Y} = \cos(\alpha)\ I + \sin(\alpha)\ iY = \begin{pmatrix} \cos(\alpha) & \text{-}\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

In the following we present an alternative, more pedestrian approach for calculating the matrix exponential (which works for more general cases and not just those which happen to be rotation matrices) using diagonalization.

Given the diagonalization of a matrix $M = U\Lambda U^{-1}$ (which we have seen in exercise 2) we can compute it's square

$$M^2 = U\Lambda \underbrace{U^{-1}U}_{=I} \Lambda U^{-1} = U\Lambda^2 U^{-1}$$

by taking the element-wise square of the eigenvalues in Λ^2 since U and it's inverse cancel out. For higher powers the matrix of eigenvectors U and it's inverse U^{-1} cancel out in a similar way.

Using the same reasoning as in the diagonal case before we can thus calculate the matrix exponential of a matrix by diagonalizing it and taking the element-wise exponential of the eigenvalues on the diagonal of Λ, so for $M = U\Lambda U^{-1}$

$$e^M = e^{U\Lambda U^{-1}} = Ue^{\Lambda}U^{-1}$$

Now we can take the Pauli matrix $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. We notice that Y is hermitian. By the spectral theorem it can diagonalized by a unitary Matrix U with real eigenvalues.

Following problem 2, we calculate the diagonalization of Y:

$$\underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{Y} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}}_{U^{-1}=U^H}$$

Here we only diagonalized the Pauli matrix Y, however an additional constant $i\alpha$ simply scales all of it's eigenvalues:

$$i\alpha Y = i\alpha U\Lambda U^H = U(i\alpha\Lambda)U^H$$

By taking the element-wise exponential of $i\alpha\Lambda$ we find that

$$e^{i\alpha Y} = Ue^{i\alpha\Lambda}U^H = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}\begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} = \begin{pmatrix} \frac{e^{i\alpha}+e^{-i\alpha}}{2} & i\frac{e^{i\alpha}-e^{-i\alpha}}{2} \\ -i\frac{e^{i\alpha}-e^{-i\alpha}}{2} & \frac{e^{i\alpha}+e^{-i\alpha}}{2} \end{pmatrix}$$

Using the Euler identity $e^{i\phi} = \cos(\phi) + i\sin(\phi)$ it is easy to show that $sin(\phi) = \frac{e^{i\phi}-e^{-i\phi}}{2i}$ and $cos(\phi) = \frac{e^{i\phi}+e^{-i\phi}}{2}$

and so the matrix exponential $e^{i\alpha Y}$ is equal to $\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ sin(\alpha) & \cos(\alpha) \end{pmatrix}$.

Problem 5 : Tensor product

- $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

  Then

  $$A \otimes B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  and

  $$B \otimes A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  are not equal.

- To prove the equivalence we can simply apply the definition of tensor product and do the calculations on both sides.

- Now we have to prove that the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

can not be written as a tensor product of two matrices $A \otimes B$. First, we have to calculate the product. Using the definition of tensor product given in the exercise session we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad .$$

Considering a matrix A of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the tensor product $A \otimes B$ has the form

$$\begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \quad .$$

Therefore we have

$$a_{12}B = a_{21}B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

that means either $B = \mathbf{0}$ or $a_{12} = a_{21} = 0$. Since B can't be the null matrix, we can conclude $a_{12} = a_{21} = 0$. For the diagonal blocks, we have that

$$a_{11}B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad a_{22}B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or, equivalently

$$a_{22} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which has no nonzero solution, hence writing the matrix as a tensor product of two matrices is not possible.

- For the last part we need to show that $|\psi\rangle = |00\rangle + |11\rangle$ cannot be written as a simple tensor product of two states.

  Taking two general states $|\phi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi_2\rangle = \gamma|0\rangle + \delta|1\rangle$ their tensor product is given by

  $$|\phi_1\rangle \otimes |\phi_2\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

  .

  Comparing the coefficients with the state $|\psi\rangle = |00\rangle + |11\rangle$ we arrive at the system of equations

  $$\begin{aligned} \alpha\gamma &= 1 \\ \alpha\delta &= 0 \\ \beta\gamma &= 0 \\ \beta\delta &= 1 \end{aligned}$$

  Clearly this has no solution and so writing $|\psi\rangle$ as a tensor product of two states is impossible.