Quantum Information and Quantum Computing, Solutions 10

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Problem 1 : A depolarising channel

(a) We are asked to find the Kraus-operator representation of

$$\mathcal{E}(\hat{\rho}) \coloneqq p\,\frac{\hat{\mathbb{1}}}{2} + (1-p)\,\hat{\rho}. \tag{1}$$

Following the hint we start by proving the equality

$$\mathbb{1} = \frac{1}{2}(\hat{\rho} + \hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z). \tag{2}$$

Plugging $\hat{\rho} = \frac{\hat{\mathbb{1}}}{2}$ into (2), it is easy to verify that

$$\frac{1}{2}(\frac{\hat{\mathbb{1}}}{2} + \hat{\sigma}_x\frac{\hat{\mathbb{1}}}{2}\hat{\sigma}_x + \hat{\sigma}_y\frac{\hat{\mathbb{1}}}{2}\hat{\sigma}_y + \hat{\sigma}_z\frac{\hat{\mathbb{1}}}{2}\hat{\sigma}_z) = \hat{\mathbb{1}} \tag{3}$$

using that $\hat{\sigma}_x\hat{\sigma}_x = \hat{\sigma}_y\hat{\sigma}_y = \hat{\sigma}_z\hat{\sigma}_z = \hat{\mathbb{1}}$.

Furthermore for the Pauli matices $\hat{\sigma}_j$, $j \in \{x, y, z\}$ we have that

$$\frac{1}{2}(\hat{\sigma}_j + \hat{\sigma}_x\hat{\sigma}_j\sigma_x + \hat{\sigma}_y\sigma_j\hat{\sigma}_y + \hat{\sigma}_z\sigma_j\hat{\sigma}_z) = 0 \tag{4}$$

which can be shown using the commutation relations of the Pauli matrices.

Now we recall that any two-dimensional density matrix can be written in the bloch representation given by

$$\rho = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{r}\cdot\hat{\boldsymbol{\sigma}}) = \frac{1}{2}(\hat{\mathbb{1}} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z), \tag{5}$$

therefore by writing the density matrix $\hat{\rho}$ in (2) in this form and using (3) and (4) we can verify the equality.

Finally using (2) to replace the identity in the definition of the channel (1) it follows that

$$\mathcal{E}(\hat{\rho}) = (1 - \frac{3}{4}p)\rho + \frac{1}{4}p(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z) \tag{6}$$

and so we find the Kraus operators
$M_1 = \sqrt{1 - \frac{3}{4}p}\,\hat{\mathbb{1}}$, $M_2 = \frac{1}{2}\sqrt{p}\,\hat{\sigma}_x$, $M_3 = \frac{1}{2}\sqrt{p}\,\hat{\sigma}_y$ and $M_4 = \frac{1}{2}\sqrt{p}\,\hat{\sigma}_z$.

(b) We apply the depolarizing channel

$$\mathcal{E}(\hat{\rho}) := p\,\frac{\hat{\mathbb{1}}}{2} + (1-p)\,\hat{\rho} \tag{7}$$

to a density matrix in the Bloch representation

$$\rho = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\hat{\boldsymbol{\sigma}}) \tag{8}$$

and find that

$$\mathcal{E}(\frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\hat{\boldsymbol{\sigma}})) = \frac{1}{2}(\mathbb{1} + (1-p)\mathbf{r}\cdot\hat{\boldsymbol{\sigma}}). \tag{9}$$

The resulting state is given again in the Bloch representation, but with the vector $\mathbf{r}$ rescaled by $1-p$. Points on a sphere with radius r are mapped to points on the smaller sphere with radius $(1-p)r$. In particular, if we apply the depolarizing channel to pure states (which are those for which $r = 1$ in the Bloch representation), for $p \neq 0$ we obtain mixed states ($r < 1$).

Problem 2 : Noise model in Qiskit

See the notebook for the hands-on part of the exercise.