Statistical Physics of Computation - Exercises

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Week 4

4.1 Tools for the replica method

4.1.1 Eigenvalues of the replica symmetric overlap

Consider an $n \times n$ matrix with all entries on the diagonal equal to q and all entries outside of the diagonal equal to r. This is called a replica symmetric (RS) matrix. We want to compute the eigenvalues of the matrix.

1. Map this problem to finding the eigenvalues of a matrix with all entries equal to r.
2. Recognise that the eigenvalues of such a matrix are either 0 with multiplicity $n-1$ or nr with multiplicity one.
3. Find the eigenvalues of the original matrix.

4.1.2 Entropic piece of the free entropy

A common ingredient of replica computations where the degrees of freedom are constrained on the sphere is given by the integral

$$S_{\text{entropy}}(q^{ab}) = \log\left(\int \prod_a \mathrm{d}\mathbf{x}^a \prod_{a\leq b} \delta\left(Nq^{ab} - (\mathbf{x}^a)^\top \mathbf{x}^b\right)\right) \tag{1}$$

for a given symmetric matrix q^{ab}, where $q^{aa} = 1$ on the diagonal. Here $a = 1, \dots, n$, and each $x^a \in \mathbb{R}^N$ In the context of the storage problem we saw in the lecture, q^{ab} is the $n \times n$ replica overlap, with $1 \leq a, b \leq n$, $\mathbf{x}^a$ and $\mathbf{x}^b$ are two replicas of the perceptron's weights, which means that they are two independent sample both extracted from the uniform distribution on the N-dimensional sphere, and N is taken to be large. This quantity encodes the different ways in which n vectors on the sphere x^a can have a certain overlap structure q^{ab}.

1. Use the Fourier representation of the delta functions to rewrite the integral as

$$S_{\text{entropy}}(q^{ab}) = \log\left(\int \mathrm{d}\hat{q}\, e^{N\sum_{a,b}\hat{q}_{ab}q_{ab}} \int \prod_a \mathrm{d}\mathbf{x}^a \exp\left\{-\sum_{a,b}\hat{q}^{ab}(\mathbf{x}^a)^\top \mathbf{x}^b\right\}\right) \tag{2}$$

2. Show that one can write

$$S_{\text{entropy}}(q^{ab}) = \log\left(\int \mathrm{d}\hat{q}\, e^{N\left[\sum_{a,b}\hat{q}_{ab}q_{ab}+I_{\text{entropy}}(\hat{q}^{ab})\right]}\right) \tag{3}$$

where

$$I_{\text{entropy}}(\hat{q}^{ab}) = \log\left(\int \prod_a \mathrm{d}x^a \exp\left\{-\sum_{a,b}\hat{q}^{ab}x^a x^b\right\}\right) \tag{4}$$

Notice that here the integral is not anymore over x is not $\mathbb{R}^N$, but it is just over $\mathbb{R}$.

3. Now we are reduced to a n-dimensional integral. Using a Gaussian integration, show that

$$I_{\text{entropy}}(\hat{q}^{ab}) = -\frac{1}{2}\log\det 2\hat{q}^{ab} + \frac{n}{2}\log(2\pi) \tag{5}$$

4. Using the saddle point method for $N \gg 1$ conclude that at leading order

$$\frac{1}{N}S_{\text{entropy}}(q^{ab}) = \frac{1}{2}\log\det q^{ab} + 2n + \frac{n}{2}\log(2\pi) \tag{6}$$

A comment on this formula: it's typical in the physics literature to drop the last two pieces because they are independent of q^{ab}, i.e. they do not alter the state equations. Hint: you can use Jacobi's formula to take the derivative of a determinant.

5. Now let's assume we are replica symmetric, with q^{ab} with entries on the diagonal equal to one and entries out of the diagonal equal to q. Neglecting all q-independent term, show that

$$\lim_{n\to 0}\frac{S_{\text{entropy}}}{nN} = \frac{1}{2}\log(1-q) + \frac{q}{2(1-q)} \tag{7}$$

4.2 Zooming in close to the SAT/UNSAT transition in the storage problem

We have seen in class that the overlap q is the solution to the state equation

$$\frac{q}{1-q} = \frac{\alpha}{2\pi}\int \mathcal{D}t\, e^{-\frac{(\kappa-\sqrt{q}t)^2}{1-q}}\left[H\left(\frac{\kappa-\sqrt{q}t}{\sqrt{1-q}}\right)\right]^{-2} \tag{8}$$

where $\mathcal{D}$ is the Normal Gaussian measure and $H(x)$ is the cumulative function of the Gaussian

$$H(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-\frac{t^2}{2}}\,\mathrm{d}t \tag{9}$$

Here κ is the margin and $\alpha = P/N$ is the number of patterns per dimension.

1. Assume that there is a phase transition at $\alpha = \alpha_c(\kappa)$. For which values of α is there a SAT phase and for which an UNSAT one? What is the effect of changing κ on the critical $\alpha_c(\kappa)$?

2. What is the overlap at small α? Answer intuitively and then check with the equation.

3. What is the overlap as α approaches α_c from the SAT phase, i.e. $\alpha \to \alpha_c^-$?

4. We now use point 3 to compute $\alpha_c(\kappa)$. It is often the case that even if we cannot solve analytically the state equations, we can at least characterise the phase transition points to some extent. Show that in the limit $q \to 1^-$, the cumulative function H has the following limit

$$H\left(\frac{\kappa - \sqrt{q}t}{\sqrt{1-q}}\right) = \theta(t-\kappa) + \theta(\kappa - t)e^{-\frac{(\kappa-t)^2}{2(1-q)}}\sqrt{\frac{1-q}{2\pi}}\frac{1}{\kappa - t} \tag{10}$$

Hint: recall that for very large arguments $x \to +\infty$, $H(x)$ has the following asymptotics

$$H(x) \approx \frac{1}{\sqrt{2\pi x}}e^{-\frac{x^2}{2}} \tag{11}$$

5. The decomposition of the H function strongly suggests us to split the integral in two regions, namely $t \in (-\infty, \kappa)$ and $t \in (\kappa, +\infty)$. Using the expansion for the H function, show that the contribution from the integral $t \in (\kappa, +\infty)$ goes to zero as $q \to 1^-$.

6. Thus, in the $q \to 1^-$ limit we can approximate

$$\int \mathcal{D}t\, e^{-\frac{(\kappa-\sqrt{q}t)^2}{1-q}} \left[H\left(\frac{\kappa - \sqrt{q}t}{\sqrt{1-q}}\right)\right]^{-2} \approx \int_{-\infty}^{\kappa} \mathcal{D}t\, e^{-\frac{(\kappa-\sqrt{q}t)^2}{1-q}} \left[H\left(\frac{\kappa - \sqrt{q}t}{\sqrt{1-q}}\right)\right]^{-2}. \tag{12}$$

Now use again the expansion for H to show that the state equation can be written as

$$\alpha = \left[\int_{-\infty}^{\kappa} \mathcal{D}t\,(\kappa - t)^2\right]^{-1} \tag{13}$$

7. Conclude that for $\kappa = 0$, $\alpha = 2$

To be precise, one should also consider a possible contribution to the integral $t \in (\kappa - \epsilon\sqrt{1-q}, \kappa + \epsilon\sqrt{1-q})$ where the expansion in point 4 breaks down, but one can show that in this problem this contribution also vanishes. This part of the computation is outside of the scope (and level of mathematical rigour) of the course, so we do not pursue it.