Statistical Physics of Computation - Exercises

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September 2024

Week 3

3.1 Lausanne, the happy city

Consider the N citizens of Lausanne as nodes in a given graph G, with vertex set V and edge set E: two people are neighbours on the graph if they know each other. They either are from UNIL of EPFL, which you can consider as a binary variable s_i, $1 \leq i \leq N$ on each of the nodes. The city is a happy one if each couple of neighbours is from different universities.

1. Identify the physical degrees of freedom of the problem, and the couplings. Recall that the degrees of freedoms are the variables in the problem you are free to modify, and the couplings are the constraints on your degrees of freedom
2. Write a function of the overall configuration $\{s_i\}_{i=1}^N$ that equals zero if the constraint are not all satisfied. It may be useful to use Kronecker delta functions.
3. Write an equation for the number of configurations of the variables $\{s_i\}_{i=1}^N$ that make the city happy.
4. We define the Garner volume for a fixed graph G as the fraction of configurations $\{s_i\}_{i=1}^N$ that satisfy the constraints. Write an expression for the Gardner volume.
5. Consider now our graph to be extracted randomly from d regular graph with N nodes. Provide an expression for the average Gardner volume.
6. Given a graph $G = (V, E)$, compute the total number of constraints and the total number of variables of this constraint satisfaction problem. In which limit do you expect to have a SAT (satisfiable) phase? In which limit do you expect to have an UNSAT (unsatisfiable) phase?
7. Now consider completely different problem: the city is happy if each pair of people that know each other (still defined by a graph G) have a difference of income of more than d. Assume that one can at most earn F francs. What are the degrees of freedom of the model, and what are the couplings? What is the entropy associated to the Gardner volume for this problem? In which regime will I have a SAT / UNSAT phase?

3.2 Perceptron learning with sign-constrained weights

Consider now the same problem as we saw in the lecture. Given P points ξ^μ in dimension N with labels $\sigma^\mu \in \{\pm 1\}$, we want to find an hyperplane J such that

$$\sigma^\mu \frac{J \cdot \xi^\mu}{\sqrt{N}} > \kappa \,. \tag{1}$$

We think of the numbers ξ_i^μ, $\mu = 1, \ldots, P$, $i = 1, \ldots, N$ as being all generated independently from the standard Gaussian $N(0, 1)$. The labels σ^μ are fixed. Also, we take J to have norm $\sqrt{N}$.

Now, suppose that we want to impose an additional constraint, i.e. we require that the coordinates of the hyperplane have fixed signs $\epsilon_i \in \{\pm 1\}$, i.e. that

$$\text{sign}(J_i) = \text{sign}(\epsilon_i) \,. \tag{2}$$

Again, think of the signs ϵ_i as fixed. This is relevant for biologically-inspired problems, where each weight is though of as a neuronal connection, and some of them are either excitatory ($\epsilon_i = +1$) or inibitory ($\epsilon_i = -1$).

1. What are the degrees of freedom of the model? And what are the couplings? Note that by degrees of freedom we means the quantities of the model you can tune to satisfy the constraint, the other quantities that define the constraint are the couplings.

2. Write the averaged entropy density associated to Gardner's volume for this problem.

3. Argue that the entropy you derived in point 1 does not depend on σ and ϵ.