Magnetism in materials

Solutions - Week 01

1. Calculate the magnetic moment of a free electron (with g = 2). What is the Larmor precession frequency of this electron in a magnetic field of flux density 0.3 T? What is the difference in energy of the electron if its spin points parallel or antiparallel to the magnetic field? Convert this energy into a frequency.

   Solution Using the classical gyromagnetic ratio for an electron, the g-factor and the angular momentum linked to a spin :

$$\begin{aligned}
\gamma &= \frac{e}{2m_e} \\
\frac{\gamma_e}{\gamma} &= g_e = 2 \\
\mu &= \gamma g_e \hbar m_s = \mu_B g_e m_s = 9.274 \cdot 10^{-24}\text{Am}^2 \\
f &= \frac{\omega_L}{2\pi} = \frac{\gamma_e B}{2\pi} = 8.4\text{Hz} \\
E &= \mu B = \mu_B g_e m_s B \\
\Delta E &= 2\mu_B g_e m_s B = 5.56 \cdot 10^{-24}\text{J}
\end{aligned}$$

   Using the relation between the photon frequency and his energy :

$$f = \frac{E}{h} = \frac{\omega_L}{2\pi}$$

2. Let's suppose that we have a magnetic moment $\boldsymbol{\mu}$ in a magnetic $\mathbf{B}$ field. The magnetic $\mathbf{B}$ field is only along the z axis ($\mathbf{B} = B\hat{\mathbf{e}}_z$) and the magnetic moment $\boldsymbol{\mu}$ is initially at an angle of θ to $\mathbf{B}$ and in the xz plane. Found the time dependant expression for $\boldsymbol{\mu}$ and the Larmor frequency (ω_L).

   Solution The torque applied on the magnetic moment can be written as :

$$\tau = \boldsymbol{\mu} \times \mathbf{B} = \frac{dL}{dt}$$

   And using the gyromagnetic constant :

$$\begin{aligned}
\boldsymbol{\mu} &= \gamma \mathbf{L} \\
\frac{d\boldsymbol{\mu}}{dt} &= \gamma \boldsymbol{\mu} \times \mathbf{B}
\end{aligned}$$

Which can be written component by component :

$$\dot{\mu}_x = \gamma B \mu_y$$
$$\dot{\mu}_y = -\gamma B \mu_x$$
$$\dot{\mu}_z = 0$$

Using the initial condition, the solution is then :

$$\mu_x = -\mu \sin\theta \cos(\omega_L t)$$
$$\mu_y = \mu \sin\theta \sin(\omega_L t)$$
$$\mu_z = \mu \cos\theta$$

With the Larmor frequency $\omega_L = \gamma B$.

3. Using the definition of spin operators :

$$\hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to prove that :

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hat{S}_z$$

With all the cyclic permutation (x, y, z) and the two following :

$$\left[\hat{\mathbf{S}}^2, \hat{S}_z\right] = 0$$
$$\hat{\mathbf{S}}^2 \left|\psi\right\rangle = \frac{3}{4}\left|\psi\right\rangle$$

With $\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ and $\left|\psi\right\rangle$ an arbitrary state.

Solution By simple matrix multiplications, it is possible to prove the identity. We can remark that the matrix $\hat{S}_i = \frac{1}{2}\hat{\sigma}_i$ with $i = x, y, z$ and $\hat{\sigma}_i$ the corresponding Pauli's matrices. It is then clear that :

$$\hat{S}_i^2 = \frac{1}{4}\hat{I}$$

With $\hat{I}$ the identity matrix.

4. Prove that :

$$\left[\hat{S}_+, \hat{S}_-\right] = 2\hat{S}_z$$
$$\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+ = 2\left(\hat{S}_x^2 + \hat{S}_y^2\right)$$

With $\hat{S}_+$ and $\hat{S}_-$ the raising and lowering operator.

Solution

$$\begin{aligned}\left[\hat{S}_+,\hat{S}_-\right] &= \left(\hat{S}_x+i\hat{S}_y\right)\left(\hat{S}_x-i\hat{S}_y\right)-\left(\hat{S}_x-i\hat{S}_y\right)\left(\hat{S}_x+i\hat{S}_y\right)\\ &= i\left[\hat{S}_y,\hat{S}_x\right]-i\left[\hat{S}_x,\hat{S}_y\right]\\ &= i\left(-i\hat{S}_z\right)-i\left(i\hat{S}_z\right)\\ &= 2\hat{S}_z\\ \hat{S}_+\hat{S}_-+\hat{S}_-\hat{S}_+ &= \left(\hat{S}_x+i\hat{S}_y\right)\left(\hat{S}_x-i\hat{S}_y\right)+\left(\hat{S}_x-i\hat{S}_y\right)\left(\hat{S}_x+i\hat{S}_y\right)\\ &= 2\left(\hat{S}_x^2+\hat{S}_y^2\right)\end{aligned}$$

5. Using the previous exercise and the following commutation rules

$$\begin{aligned}\left[\hat{\mathbf{S}}^2,\hat{S}_\pm\right] &= 0\\ \left[\hat{S}_z,\hat{S}_\pm\right] &= \pm\hat{S}_\pm\end{aligned}$$

prove that

$$\hat{S}_\pm\left|S,S_z\right\rangle = \sqrt{S(S+1)-S_z(S_z\pm1)}\left|S,S_z\pm1\right\rangle$$

Where $\left|S,S_z\right\rangle$ represents a state with total spin angular momentum $\hbar^2S(S+1)$ and z component of spin angular momentum $\hbar S_z$ which is equivalent to say :

$$\begin{aligned}\hat{\mathbf{S}}^2\left|S,S_z\right\rangle &= S(S+1)\left|S,S_z\right\rangle\\ \hat{S}_z\left|S,S_z\right\rangle &= S_z\left|S,S_z\right\rangle\end{aligned}$$

Solution At first, we prove that $\hat{S}_\pm\left|S,S_z\right\rangle$ is a eigenstate of $\hat{S}_z$ and $\hat{\mathbf{S}}^2$ and then compute the relative eigenvalue.

Since $\hat{\mathbf{S}}^2$ and $\hat{S}_\pm$ commute :

$$\begin{aligned}\hat{\mathbf{S}}^2\hat{S}_\pm\left|S,S_z\right\rangle &= \hat{S}_\pm\hat{\mathbf{S}}^2\left|S,S_z\right\rangle\\ &= S(S+1)\hat{S}_\pm\left|S,S_z\right\rangle\end{aligned}$$

Using the commutation rule between $\hat{S}_z$ and $\hat{S}_\pm$:

$$\begin{aligned}\hat{S}_z\hat{S}_\pm\left|S,S_z\right\rangle &= \left(\hat{S}_\pm\hat{S}_z\pm\hat{S}_\pm\right)\left|S,S_z\right\rangle\\ &= \left(S_z\pm1\right)\hat{S}_\pm\left|S,S_z\right\rangle\end{aligned}$$

All what is left now is to proceed to the normalization of the state. For this we use the previous exercise that lead us to :

$$\hat{S}_\pm\hat{S}_\mp = \hat{\mathbf{S}}^2-\hat{S}_z^2\pm\hat{S}_z$$

Then the normalization go as follow :

$$\begin{aligned}\|\hat{S}_\mp\left|S,S_z\right\rangle\|^2 &= \langle S,S_z|\hat{S}_\pm\hat{S}_\mp|S,S_z\rangle\\ &= \langle S,S_z|\hat{\mathbf{S}}^2-\hat{S}_z^2\pm\hat{S}_z|S,S_z\rangle\\ &= S(S+1)-S_z(S_z\pm1)\end{aligned}$$

6. Indicating with $\sigma=(\hat{\sigma}_x,\hat{\sigma}_y,\hat{\sigma}_z)$ the Pauli matrices, show that for any $\mathbf{r}$ such that $|\mathbf{r}|=1$

$$\exp\{i\alpha\mathbf{r}\cdot\sigma\} = \hat{\mathbf{1}}_2\cos\alpha+i\mathbf{r}\cdot\sigma\sin\alpha$$

solution we start by showing that $(\mathbf{r}\cdot\sigma)^2 = \hat{\mathbf{1}}_2$

$$
\begin{aligned}
(\mathbf{r}\cdot\sigma)^2 &= r_x^2\hat{\sigma}_1^2 + r_y^2\hat{\sigma}_2^2 + r_z^2\hat{\sigma}_3^2 + r_x r_y\{\hat{\sigma}_1,\hat{\sigma}_2\} + r_x r_z\{\hat{\sigma}_1,\hat{\sigma}_3\} + r_z r_y\{\hat{\sigma}_3,\hat{\sigma}_2\} \\
&= (r_x^2 + r_y^2 + r_z^2)\hat{\mathbf{1}}_2 = \hat{\mathbf{1}}_2
\end{aligned}
$$

Where $\{\cdot,\cdot\}$ denotes the anticommutator ans we used the propriety $\{\hat{\sigma}_i,\hat{\sigma}_j\} = 2\delta_{i,j}\hat{\mathbf{1}}_2$. So we get

$$
\begin{aligned}
\exp\{i\alpha\mathbf{r}\cdot\sigma\} &= \sum_{n=0}^{\infty}\frac{(i\alpha\mathbf{r}\cdot\sigma)^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(i\alpha\mathbf{r}\cdot\sigma)^{2n}}{2n!} + \sum_{n=0}^{\infty}\frac{(i\alpha\mathbf{r}\cdot\sigma)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n}}{2n!}\hat{\mathbf{1}}_2 + i\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n+1}}{(2n+1)!}\mathbf{r}\cdot\sigma \\
&= \hat{\mathbf{1}}_2\cos\alpha + i\mathbf{r}\cdot\sigma\sin\alpha
\end{aligned}
$$

7. Using the basis $(\left|\uparrow\uparrow\right\rangle, \left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle, \left|\downarrow\downarrow\right\rangle)$, it is possible to construct matrix representations of operators such as $\hat{\mathbf{S}}^a\cdot\hat{\mathbf{S}}^b$ remembering that, for example, an operator such as $\hat{S}_z^a$ only operates on the part of the wave function connected with the first spin. Thus we have

$$
\hat{S}_z^a = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}
$$

$$
\hat{S}_z^b = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}
$$

Construct similar representations for $\hat{S}_x^a$, $\hat{S}_x^b$, $\hat{S}_y^a$ and $\hat{S}_y^b$ and hence show that

$$
\hat{\mathbf{S}}^a\cdot\hat{\mathbf{S}}^b = \frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0\\ 0 & 2 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}
$$

Find the eigenvalues and eigenvectors of this operator.

Solution

$$\hat{S}_x^a = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{S}_x^b = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\hat{S}_y^b = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\hat{S}_y^b = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

Eigenstates	$\|\uparrow\uparrow\rangle$	$\frac{\|\uparrow\downarrow\rangle+\|\downarrow\uparrow\rangle}{\sqrt{2}}$	$\|\downarrow\downarrow\rangle$	$\frac{\|\uparrow\downarrow\rangle-\|\downarrow\uparrow\rangle}{\sqrt{2}}$
Eigenvectors	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$
Eigenvalues	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$

8. Show that the operator

$$\hat{S}_{\theta,\phi} = \sin\theta\cos\phi\hat{S}_x + \sin\theta\sin\phi\hat{S}_y + \cos\theta\hat{S}_z$$

which represents the spin operator for the component of spin along a direction determined by the spherical polar angles θ and ϕ,has eigenvalue $\pm\frac{1}{2}$ and eigenstates of the form

$$|\uparrow\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{i\phi} \end{pmatrix}$$

$$|\downarrow\rangle = \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2e^{i\phi} \end{pmatrix}$$

show further that

$$\hat{S}_{\theta\phi}^2 = \frac{1}{4}\hat{\mathbf{1}}_2$$

with $\hat{\mathbf{1}}_2$ the $2x2$ identity matrix.

Solution In the base of $\hat{S}_z$ eigenstates

$$\hat{S}_{\theta,\phi} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix}$$

So the zeros of its characteristic polynomial are found from

$$\begin{aligned}\left(\frac{\cos\theta}{2} - \lambda\right)\left(-\frac{\cos\theta}{2} - \lambda\right) - \frac{\sin^2(\theta)}{4}e^{i\phi}e^{-i\phi} = 0 \\ \lambda^2 - \frac{1}{4}(\cos^2\theta + \sin^2\theta) = 0 \\ \lambda = \pm\frac{1}{2}\end{aligned}$$

with a direct calculation we can verify that

$$\begin{aligned}\hat{S}_{\theta\phi}\left|\uparrow\right\rangle &= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{i\phi} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta\cos\theta/2 + \sin\theta\sin\theta/2 \\ \sin\theta\cos\theta/2e^{i\phi} - \cos\theta\sin\theta/2e^{i\phi} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (\cos\theta/2^2 - \sin\theta/2^2)\cos\theta/2 + 2\sin^2\theta/2\cos\theta/2 \\ e^{i\phi}[2\sin\theta/2\cos\theta/2^2 - (\cos^2\theta/2 - \sin^2\theta/2)\sin\theta/2] \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta/2(\cos^2\theta/2 + \sin^2\theta/2) \\ \sin\theta/2e^{i\phi}(\cos^2\theta/2 + \sin^2\theta/2) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{i\phi} \end{pmatrix} = \frac{1}{2}\left|\uparrow\right\rangle\end{aligned}$$

and a similar calculation shows $\hat{S}_{\theta\phi}\left|\downarrow\right\rangle = -\frac{1}{2}\left|\downarrow\right\rangle$.
Calculating $\hat{S}^2_{\theta\phi}$

$$\begin{aligned}\hat{S}^2_{\theta\phi} &= \frac{1}{4}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}^2 \\ &= \frac{1}{4}\begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta e^{-i\phi} - \cos\theta\sin\theta e^{-i\phi} \\ \cos\theta\sin\theta e^{i\phi} - \cos\theta\sin\theta e^{i\phi} & \sin^2\theta + \cos^2\theta \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

9. If the magnetic field $\mathbf{B}$ is uniform in space, show that this is consistent with writing $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$ and show that $\nabla\cdot\mathbf{A} = 0$. Are there other choices of $\mathbf{A}$ that would produce the same $\mathbf{B}$?

solution Without loss of generality we can assume that $\mathbf{B} = B\hat{\mathbf{z}}$, so that

$$\begin{aligned} A &= \frac{1}{2}\mathbf{B} \times \mathbf{r} \\ &= \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} \frac{-yB}{2} \\ \frac{xB}{2} \\ 0 \end{pmatrix} \end{aligned}$$

Calculating $\nabla \times \mathbf{A}$

$$\nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_x \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$$

And $\nabla \cdot \mathbf{A} = 0$

$$\nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z = 0 + 0 + 0 = 0$$

For the Gauge freedom, any $\mathbf{A}' = \mathbf{A} + \nabla\phi(\mathbf{r})$, with $\phi(\mathbf{r})$ a derivable function, will give the same $\mathbf{B}$.

10. Using the propriety of the Pauli matrices $\sigma = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$

$$(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b})$$

with $\mathbf{a}, \mathbf{b}$ vectors, show that the kinetic energy operator for an electron $\frac{\hat{\mathbf{p}}^2}{2m}$ can be rewritten as

$$\frac{(\sigma \cdot \hat{\mathbf{p}})^2}{2m}$$

If a magnetic field is applied one must replace $\hat{\mathbf{p}}$ by $\hat{\mathbf{p}} + e\mathbf{A}$ show that this replacement substituted into the previous result leads to kinetic energy of the form

$$\frac{(\hat{\mathbf{p}} + e\mathbf{A})^2}{2m} + g\mu_B \mathbf{B} \cdot \hat{\mathbf{S}}$$

where the g-factor in this case is $g = 2$. (Note: $\hat{\mathbf{p}}$ s an operator and will not commute with $\mathbf{A}$)

solution using the commutation relations

$$[\hat{p}_\alpha, \hat{p}_\beta] = 0 \quad \alpha, \beta \in \{x, y, z\}$$
$$\{\hat{\sigma}_i, \hat{\sigma}_i\} = 2\delta_{i,j}\hat{\mathbf{1}}_2 \quad i, j \in \{1, 2, 3\}$$

One can calculate

$$\begin{aligned}(\sigma \cdot \hat{\mathbf{p}})^2 &= (\hat{\sigma}_1\hat{p}_x + \hat{\sigma}_2\hat{p}_y + \hat{\sigma}_3\hat{p}_z)^2 \\ &= \hat{\sigma}_1^2\hat{p}_x^2 + \hat{\sigma}_2^2\hat{p}_y^2 + \hat{\sigma}_3^2\hat{p}_z^2 + \{\hat{\sigma}_1, \hat{\sigma}_2\}\hat{p}_x\hat{p}_y + \{\hat{\sigma}_1, \hat{\sigma}_3\}\hat{p}_x\hat{p}_z + +\{\hat{\sigma}_2, \hat{\sigma}_3\}\hat{p}_z\hat{p}_y \\ &= \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = \hat{\mathbf{p}}^2\end{aligned}$$

replacing $\hat{\mathbf{p}}$ by $\hat{\mathbf{p}} + e\mathbf{A}$

$$\begin{aligned}\frac{[\sigma \cdot (\hat{\mathbf{p}} + e\mathbf{A})]^2}{2m} &= (2m)^{-1}[(\sigma \cdot \hat{\mathbf{p}})^2 + e(\sigma \cdot \hat{\mathbf{p}})(\sigma \cdot \mathbf{A}) + e(\sigma \cdot \mathbf{A})(\sigma \cdot \hat{\mathbf{p}}) + e^2(\sigma \cdot \mathbf{A})^2] \\ &= (2m)^{-1}[\hat{\mathbf{p}}^2 + \mathbf{A}^2 + e\hat{\mathbf{p}} \cdot \mathbf{A} + e\mathbf{A} \cdot \hat{\mathbf{p}} + ie\sigma \cdot (\hat{\mathbf{p}} \times \mathbf{A}) + ie\sigma \cdot (\mathbf{A} \times \hat{\mathbf{p}})] \\ &= (2m)^{-1}[(\hat{\mathbf{p}} + e\mathbf{A})^2 + ie\sigma(\mathbf{A} \times \hat{\mathbf{p}} + \hat{\mathbf{p}} \times \mathbf{A})]\end{aligned}$$

To show that the second term correspond to the Zeeman term consider its action on a wave function ψ

$$\begin{aligned}\frac{ie}{2m}\sigma \cdot (\mathbf{A} \times \hat{\mathbf{p}} + \hat{\mathbf{p}} \times \mathbf{A})\psi &= \frac{\hbar e}{2m}\sigma \cdot [\mathbf{A} \times \nabla\psi + \nabla \times (\mathbf{A}\psi)] \\ &= \mu_B g\hat{\mathbf{S}} \cdot [\mathbf{A} \times \nabla\psi + (\nabla \times \mathbf{A})\psi - \mathbf{A} \times \nabla\psi] \\ &= \mu_B g\hat{S} \cdot \mathbf{B}\psi\end{aligned}$$

with $g = 2$; $\hat{\mathbf{S}} = \sigma/2$ and we used the identity $\nabla \times (\mathbf{A}\psi) = (\nabla \times \mathbf{A})\psi - \mathbf{A} \times \nabla\psi$.

11. An electron in a magnetic field aligned along the z-direction has a Hamiltonian (energy) operator

$$\hat{\mathcal{H}} = g\mu_B B\hat{S}_z$$

The time-dependent Schrödinger equation states that

$$\hat{\mathcal{H}}\psi(t) = i\hbar\frac{d\psi}{dt}$$

So that

$$\psi(t) = \exp\{-i\hat{\mathcal{H}}t/\hbar\}\psi(0)$$

Using the result from ex. 6 show that $\psi(t)$ written as a spinor is

$$\psi(t) = \begin{pmatrix} \exp\{-ig\mu_B Bt/2\hbar\} & 0 \\ 0 & \exp\{ig\mu_B Bt/2\hbar\} \end{pmatrix} \psi(0)$$

and using the results from ex. 8 show that this corresponds to the evolution of the spin state in such a way that the expected value of θ is conserved but ϕ rotates with an angular frequency given by $geB/2m$. This demonstrates that the phenomenon of Larmor precession can also be derived from a quantum mechanical treatment.

solution Using the identity $\exp\{i\alpha\hat{\sigma}_z\} = \hat{\mathbf{1}}_2 \cos\alpha + i\hat{\sigma}_z \sin\alpha$, and writing the initial condition in the form

$$\psi(0) = \begin{pmatrix} \cos(\theta_0/2) \\ \sin(\theta_0/2)e^{i\phi_0} \end{pmatrix}$$

one gets

$$\begin{aligned}
\psi(t) &= \exp\{-i\hat{\mathcal{H}}t/\hbar\}\psi(0) \\
&= \exp\{-ig\mu_B Bt/2\hbar\hat{\sigma}_z\}\psi(0) \\
&= [\hat{\mathbf{1}}_2 \cos(g\mu_B Bt/2\hbar) + i\hat{\sigma}_z \sin(g\mu_B Bt/2\hbar)]\psi(0) \\
&= \begin{pmatrix} \cos(g\mu_B Bt/2\hbar) + i\sin(g\mu_B Bt/2\hbar) & 0 \\ 0 & \cos(g\mu_B Bt/2\hbar) - i\sin(g\mu_B Bt/2\hbar) \end{pmatrix} \psi(0) \\
&= \begin{pmatrix} \exp\{-ig\mu_B Bt/2\hbar\} & 0 \\ 0 & \exp\{ig\mu_B Bt/2\hbar\} \end{pmatrix} \psi(0) \\
&= \begin{pmatrix} e^{-ig\mu_B Bt/2\hbar}\cos(\theta_0/2) \\ \sin(\theta_0/2)e^{i\phi_0 + ig\mu_B Bt/2\hbar} \end{pmatrix}
\end{aligned}$$

Adjusting the phase so that the first component is real

$$\psi(t) = \begin{pmatrix} \cos(\theta_0/2) \\ \sin(\theta_0/2)e^{i\phi_0 + ig\mu_B Bt/\hbar} \end{pmatrix}$$

As shown in ex. 8 this correspond to the $+\frac{1}{2}$ eigenstate of the spin operator pointing in the direction with polar angles θ_0 and $\phi_0 + g\mu_B Bt/\hbar$. So the expectation value of the spin along the field direction is conserved while the expectation value of the component in the orthogonal plane rotates with angular frequency $g\mu_B B/\hbar = geB/2m_e$