Question 8.1: The β function

In Section 4.4 we presented the idea that coupling constants run, that is, that higher order corrections can be absorbed into the definition of the coupling constants. Consider, for example, the cross section for $e^+e^- \to \mu^+\mu^-$. In the CM frame and to leading order, this is given by (up to normalization factors)

$$\sigma \propto \frac{\alpha^2}{E^2}, \tag{5.31}$$

where E is the energy of the electron. Higher order effects change this result. Most of the effect can be absorbed into the running of α, that is

$$\sigma \propto \frac{\alpha(\mu)^2}{E^2}, \tag{5.32}$$

where $\mu \sim E$ is the energy scale in the problem, and $\alpha(\mu)$ is a running coupling constant that satisfies a differential equation:

$$\frac{\partial \alpha}{\partial \log(\mu)} = \beta(\alpha). \tag{5.33}$$

The beta function can be calculated to the desired precision in perturbation theory. In QED at one loop, with only electrons in the loop, we have

$$\beta(\alpha) = B\alpha^2, \qquad B = \frac{2}{3\pi}. \tag{5.34}$$

This equation is valid for $\mu > m_e$.

1. Verify that the solution of the beta function equation is

$$\frac{1}{\alpha(\mu_1)} = \frac{1}{\alpha(\mu_2)} + B \log\left(\frac{\mu_2}{\mu_1}\right). \tag{5.35}$$

2. Use $\alpha(m_e) \approx 1/137.0$ to calculate $\alpha(m_Z)$, where $m_Z \approx 91$ GeV.

The fact that the coupling constant becomes larger for higher energy scale raises the possibility that it diverges at some scale, which would be an indication that the theory breaks down. The energy scale at which it diverges is called the Landau pole. That is, the Landau pole is $\mu_{\rm LP}$ where $\alpha(\mu_{\rm LP}) \to \infty$.

3. Find the Landau pole for α. How does it stand with respect to the Planck scale, which constitutes an upper bound on the cut-off scale of all QFT?

Measurements find that $\alpha(m_Z) \approx 1/128$. The reason for the disagreement with your result above is that there are other particles in the loop besides the electron. The generalization of Eq. (5.35) to that case is

$$\frac{1}{\alpha(\mu_1)} = \frac{1}{\alpha(\mu_2)} + B \sum_i Q_i^2 N_C^i \log\left(\frac{\max(m_i, \mu_2)}{\mu_1}\right), \tag{5.36}$$

where $N_C^i = 1(3)$ for leptons (quarks), and Q_i is the electric charge. The sum is over all the charged particles with mass below μ_1.

4. Use the physical masses and charges of the known fermions,

$$\begin{aligned} q = -1: \ & m_{e,\mu,\tau} \approx (0.5, 100, 1777)\ \text{MeV}, \\ q = +2/3: \ & m_{u,c,t} \approx (0.3, 1.4, 174)\ \text{GeV}, \\ q = -1/3: \ & m_{d,s,b} \approx (0.3, 0.4, 4.2)\ \text{GeV}, \end{aligned} \tag{5.37}$$

to calculate $\alpha(m_Z)$. How close is your result to the measured value? (Note that the values of m_u, m_d and m_s are much larger than the values quoted in PDG. The reason is that we use the "valence quark masses" rather than "running quark masses", which is what the PDG quotes. We discuss this point in Chapter 10.)

We now move to QCD. The beta function is given by $\beta(\alpha) = B\alpha_s^2$ with

$$B = -\left(11 - \frac{2n_f}{3}\right)\frac{1}{2\pi}, \tag{5.38}$$

where n_f is the number of quark flavors with masses below the relevant scale. Below we use the input $\alpha_s(m_Z) \approx 0.12$.

5. The sign of the beta function depends on the number of flavors. How many flavors are needed to change the sign of the beta function? We denote this number as N_{Cri}.

6. Sketch the shape of the function $\alpha_s(\mu)$ for μ between 1 and 10^4 GeV for (i) a theory with $n_f < n_{Cri}$, and (ii) a theory with $n_f > n_{Cri}$. Use log scale for μ.

7. Estimate Λ_{QCD}, that is, the scale where α_s =1..For simplicity, you can neglect all quark masses except m_t.

8. The proton mass is roughly $m_p \approx 3\Lambda_{\text{QCD}}$. Can you tell if the mass of the proton would be lighter or heavier if we did not have the third generation, assuming the same measured value of $\alpha_s(m_Z)$?

Question 8.2: Using the PDG even more

Read the quark model review from the PDG. Answer the following questions using the data from the light meson summary table.

1. Explain what P, C, J, I, and G stand for. For each of these QNs, indicate if they are (i) exact in Nature; or (ii) exact in QCD and QED; or (iii) approximately conserved in QCD.

2. Find the mass, width, and the above mentioned QNs of the π^0, η, ρ^0, and ω.

3. Find the Branching Ratios (BRs) of the η, ρ^0, and ω decays to two pions and to three pions.

4. From the answer to item 3, it is evident that the η does not decay to two pions, the ω decay rate to two pions is highly suppressed, while the ρ decays dominantly to two pions. Based on the QNs listed in item 2, explain these results.

Question 8.3: Leptonic pion decay

Consider the purely leptonic pion decays

$$\pi^+ \to \ell^+ \nu_\ell. \tag{10.56}$$

1. Based on the masses of the relevant particles, what final state leptons are allowed?

2. Draw the diagram for this decay. To take into account that the initial state is a pion, we need to write the amplitude using the relevant hadronic matrix element. Using Eq. (10.14),

$$\langle 0|A^\mu|\pi\rangle \equiv -i f_\pi p_\pi^\mu, \tag{10.14}$$

   show that the amplitude is given by

$$\mathcal{A}_{\pi\to\ell\nu} = -\frac{g^2}{4m_W^2} V_{ud} f_\pi p_\pi^\mu \left[\bar{u}_\ell \gamma_\mu P_L v_\nu\right], \tag{10.57}$$

   where u and v are the standard notation for the spinors and P_L is a projection operator defined in Eq. (1.3). Explain why you can approximate the W propagator as $1/m_W^2$.

$$\psi_R = P_R\psi \equiv \frac{1+\gamma_5}{2}\psi, \qquad \psi_L = P_L\psi \equiv \frac{1-\gamma_5}{2}\psi. \tag{1.3}$$

3 Using the previous result, we find that the decay rate for massless neutrinos is

$$\Gamma(\pi \to \mu\nu) = \frac{G_F^2 |V_{ud}|^2}{8\pi} m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 f_\pi^2. \tag{10.58}$$

Then , the rate between leptons is

$$R \equiv \frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 . \tag{Q10.16}$$

For the rate for massive neutrinos is

$$R = \frac{m_\pi^2\,(m_e^2 + m_1^2) - (m_e^2 - m_1^2)^2}{m_\pi^2(m_\mu^2 + m_2^2) - (m_\mu^2 - m_2^2)^2} \sqrt{\frac{[m_\pi^2 - (m_e + m_1)^2]\,[m_\pi^2 - (m_e - m_1)^2]}{[m_\pi^2 - (m_\mu + m_2)^2]\,[m_\pi^2 - (m_\mu - m_2)^2]}}, \tag{Q10.21}$$

Assuming that $m_1^2 \ll m_e^2$ and
that $m_2^2 \ll m_\mu^2$, expanding the ratio R to first order in m_1^2/m_e^2 and in m_2^2/m_μ^2,

$$\frac{R}{R_0} = 1 + \left(\frac{m_1^2}{m_e^2}\right) a_e - \left(\frac{m_2^2}{m_\mu^2}\right) a_\mu \tag{10.61}$$

where R_0 is the value of R for massless neutrinos. The expression for a_e and a_μ.

$$a_e = \frac{m_\pi^4 - 3m_e^4}{(m_\pi^2 - m_e^2)^2}, \qquad a_\mu = \frac{m_\pi^4 - 3m_\mu^4}{(m_\pi^2 - m_\mu^2)^2}.$$

4 Find the numerical values of a_e and a_μ and show that they are both positive. Argue that there is a "flat direction," *i.e.* a curve in the $(m_1,\ m_2)$ plane where $R = R_0$.

The experimental data give

$$R = (1.2327 \pm 0.0023) \times 10^{-4}. \tag{10.62}$$

Using Eq. (10.60) would give the theory prediction of $R_0 = 1.2833 \times 10^{-4}$. However, to obtain the bound, we have to take into account higher order corrections, which shift the result:

$$R_0 = (1.2352 \pm 0.0001) \times 10^{-4}. \tag{10.63}$$

Working to 2σ, derive numerical bounds on m_1 for $m_2 = 0$, and on m_2 for $m_1 = 0$. Explain why the bound on m_1 is much stronger than the one on m_2. For each of the two cases check if the leading order expansion that was used is valid. (Note: these bounds are not very strong compared to other bounds on neutrino masses, see Section 14.B.)