Flavour Physics: Week: VI

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Problem 1. The four generation SM

Consider an extension of the SM with an additional (4th) generation of quarks and leptons:

$$L_4(1,2)_{-1/2}\,, \quad E_4(1,2)_{-1/2}\,, \quad Q_4(3,2)_{+1/6}\,, \quad U_4(3,1)_{+2/3}\,, \quad D_4(3,1)_{-1/3}\,. \tag{1}$$

We denote the fourth generation mass eigenstates by t', b', τ', and ν'_τ in an obvious notation.

(a) Write down the Yukawa interactions. Use matrix notation, *e.g.*, use L_i where $i = 1, 2, 3, 4$ to denote the lepton doublets.

(b) What is the global symmetry of the model? What is the number of physical parameters related to the Yukawa terms? Separate them into masses, mixing angles and phases. (Hint: Read Sec. 7.5.3 Parameter counting in Yuval's book for further information).

(c) Is CP violated in the quark sector? Is it violated in the lepton sector?

Solution

(a) The Yukawa part of the lagrangian is

$$\mathcal{L}_{\text{Yuk}} = \lambda^E_{ij}(\overline{L_L})_i(\overline{E_R})_j\phi + \lambda^D_{ij}(\overline{Q_L})_i(\overline{D_R})_j\phi + \lambda^U_{ij}(\overline{Q_L})_i(\overline{U_R})_j\tilde{\phi} + \text{h.c.}\,. \tag{2}$$

where i, j are flavour indeces and $\tilde{\phi} = i\sigma_2\phi^*$.

(b) In general, the initial symmetry (the symmetry due to the kinematic term alone) is $[U(N_g)]^n = [SU(N_g) \times U(1)]^n$, where N_g is the number of generations, and n is the number of different states we have. In our case, we have $N_g = 4$ (four generations), and $n = 5$ (charged leptons, neutrinos, left-handed quarks, right-handed up-quarks and right-handed down-quarks). A $U(N)$ symmetry has $N_g \times N_g$ symmetry generators, and, thus, our initial $[U(4]^5$ symmetry has $(4 \times 4) \times 5 = 80$ symmetry generators. Alternatively, we could also use that $U(1)$ has only one generator, and $SU(N_g)$ has $N_g^2 - 1$ generators. Equivalently, $[SU(4) \times U(1)]^5$ has $((4\times4-1)+1)\times = 80$ generators.

The initial symmetry is broken due to the Yukawa matrices down to $U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau \times U(1)_{\tau'}$ (baryon number and N_g lepton flavours). This symmetry is $[U(1)]^5$ and has 5 generators. Therefore, we have broken $80 - 5 = 75$ symmetry generators.

We compute the number of physical parameters as $N_{\text{phys}} = N_{\text{total}} - N_{\text{broken}}$ with $N_{\text{broken}} = 75$ as previously computed. We need to compute the total number of parameters of our model.

We have three Yukawa matrices (λ^E, λ^D and λ^U). As we have four generations, they are 4×4 matrices. The total number of parameters is $3_{\text{number of matrices}}(4 \times 4)_{\text{parameters per matrix}} = 96$. Therefore, the number of physical parameters is $96 - 75 = 21$.

To identify the parameters we need to consider that the initial amount of parameters is equally divided between real and imaginary parts: $96 = 48 + 48$. And A unitary $N_g \times N_g$ matrix has $N_g(N_g - 1)/2$ real parameters and $N_g(N_g + 1)/2 - 1$ phases. These are the ones that are broken. Using $N_g = 4$ and that initially we have 5 matrices, we have $N^{\text{real}}_{\text{broken}} = 5[4(4-1)/2] = 30$ broken real parameters, and $N^{\text{phases}}_{\text{broken}} = 5[4(5)/2 - 1] = 45$ broken phases. Recovering our initial 48 real and 48 imaginary parameters we have:

- $N^{\text{real}}_{\text{phys}} = 48 - 30 = 18$, which are divided in 12 charged masses, and the rest (6) are mixing angles.
- $N^{\text{real}}_{\text{phys}} = 48 - 45 = 3$ phases. These are CP phases for the quark sector.

(c) CP is violated in the quark sector, as the 4×4 mixing matrix has three irremovable phases.

CP is not violated in the lepton sector: Since the neutrinos are degenerate, there is no lepton mixing and thus no CP violating phases.

Problem 2. (a) Derive the Wolfstein notation $\mathcal{O}(\lambda^3)$

$$V_{\text{CKM}} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{3}$$

using the CKM matrix as a function of the mixing angles and the CP phase

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{4}$$

and the equalities

$$s_{12} = \lambda\,, \qquad s_{23} = A\lambda^2\,, \qquad s_{13}e^{-i\delta} = A\lambda^3(\rho - i\eta), \tag{5}$$

Hint: Use Taylor expansion to express the elements c_{ij} as a function of the elements s_{ij}.

(b) Using the definition of the J_{CKM}

$$\mathcal{I}m(V_{ij}V_{kl}V_{il}^*V_{kj}^*) = J_{CKM}\sum_{m,n=1}^{3}\epsilon_{ikm}\epsilon_{jln}\,, \qquad (i,j,k,l = 1,2,3) \tag{6}$$

show that

$$J_{CKM} = \mathcal{I}m(V_{ud}V_{cs}V_{us}^*V_{cd}^*) \tag{7}$$

Problem 3. In the main text, we discuss the UT, that is the *bd* triangle,

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \tag{8}$$

. We mention that each of its sides is of $\mathcal{O}(\lambda^3)$, and each of its angles is of $\mathcal{O}(1)$. Here we discuss another unitarity triangle, the *bs* triangle.

(a) Write the equation for the *bs* UT.

(b) Draw the rescaled *bs* UT and label the angles as α_s, β_s and γ_s, in a similar way to our discussion of the rescaled *bd* triangle.

(c) Estimate, in terms of powers of λ, the size of each side of the *bs* UT.

Problem 4. Semileptonic decays and the CKM matrix.

A semileptonic decay is one where the final state has both hadrons and leptons. Here we consider semileptonic b decays.

(a) Draw the tree-level diagram for $\mathrm{b} \to \mathrm{ue}\overline{\nu}$ and estimate the size of the amplitude.

(b) Estimate the ratio $\frac{\Gamma(\mathrm{b}\to\mathrm{ue}\overline{\nu})}{\Gamma(\mathrm{b}\to\mathrm{ce}\overline{\nu})}$ as a function of CKM matrix elements. Neglect m_e and use $m_c/m_b \simeq 0.3$, and $m_u/m_b \simeq 0.002$. The phase space function for a three-body decay with one massive and two massless final particles is given by

$$f(x_f) = 1 - 8x_f + 8x_f^3 - x_f^4 - 12x_f^2 \log x_f \tag{9}$$

(c) The experimental value is $\frac{\Gamma(\mathrm{b}\to\mathrm{ue}\overline{\nu})}{\Gamma(\mathrm{b}\to\mathrm{ce}\overline{\nu})} \simeq 2 \times 10^{-2}$. Estimate the relevant ratio of CKM matrix elements. Comment on the agreement between your estimate and the best fit value for that ratio. (Hint: You will need to look for the value of relevant CKM elements in the PDG).