Question 2.1: Accidental symmetries

In this question, we study a classical system in order to show examples of accidental symmetries. Consider a classical one-dimensional pendulum of length ℓ. The one degree of freedom can be chosen to be θ, the angle of the pendulum. Then, the Lagrangian is given by

$$L = \frac{m\ell^2\dot{\theta}^2}{2} - mg\ell(1-\cos\theta), \qquad \text{Eq. (1.1)}$$

Assuming small oscillations ($\theta \ll 1$), we can expand the potential. Keeping only terms up to second-order, we get

$$L = \frac{m\ell^2\dot{\theta}^2}{2} - \frac{mg\ell\theta^2}{2}, \qquad \text{Eq. (1.2)}$$

which is the Lagrangian of a simple harmonic oscillator. It is well known that the frequency of a simple harmonic oscillator does not depend on its amplitude. Below we aim to understand how this result is related to accidental symmetries.

1. Show that the EoM derived from the Lagrangian of Eq. (1.16) is invariant under dilation,$\theta \to \lambda\theta$, for any finite λ. (We are then saying that L of Eq. (1.16) has dilation symmetry, despite the fact that it is only the EoM that is invariant.)
2. Does the Lagrangian of Eq. (1.1) also have dilation symmetry?
3. Expand the Lagrangian of Eq. (1.1) up to $O(\theta^4)$. Show explicitly that the θ^4 term breaks the dilation invariance. Explain why this implies that this symmetry is accidental.
4. Without a formal proof, argue that dilation symmetry implies that the frequency cannot depend on the amplitude.

What we have shown is that the dilation symmetry is accidental and that it is broken by higher order terms.

Question 2.2: Algebra

1. Using Eqs. (2.25) and (2.27), derive Eq. (2.28).

$$D^\mu = \partial^\mu + igqA^\mu\,, \tag{2.25}$$

$$[D^\mu, D^\nu] = igqF^{\mu\nu}, \tag{2.27}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.28}$$

3. Consider a kinetic term of a fermion field, $i\overline{\psi}\partial\!\!\!/\psi$, and show that it is not invariant under a local transformation: $\psi \to e^{i\theta(x)}\psi$.

4. Explain why, for terms that involve scalar and/or fermion fields but do not involve derivatives, the $\theta \to \theta(x)$ substitution has no effect on the symmetry properties.

5. Show that the covariant derivative of the field, defined in Eq. (2.25), transforms in the same way as the field itself:

$$D_\mu\phi \to e^{iq\theta(x)}D_\mu\phi. \tag{2.33}$$

6. Show that Eq. (2.33) can be written equivalently as

$$D_\mu \to e^{iq\theta(x)}D_\mu e^{-iq\theta(x)}. \tag{2.34}$$

7. Show that a mass term for A_μ, that is, $m^2 A_\mu A^\mu$ is not invariant under the transformation law of Eq. (2.26).

$$A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\theta. \tag{2.26}$$

8. Show that $F^{\mu\nu}$, defined in Eq. (2.28), is gauge invariant.

9. Show that $F^{\mu\nu}\tilde{F}_{\mu\nu}$ (with $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$) is a total derivative.

Question 2.3: Chiral symmetry

The Lagrangian for a single, massless, free Dirac fermion field is given by

$$\mathcal{L} = i\overline{\psi}\,\partial\!\!\!/\,\psi\,. \tag{2.38}$$

Consider the following two transformations:

$$\psi \to e^{i\theta}\,\psi, \qquad \psi \to e^{i\theta\gamma_5}\,\psi\,. \tag{2.39}$$

1. Show that under these two transformations, the conjugate field transforms, respectively, as

$$\overline{\psi} \to \overline{\psi}\,e^{-i\theta}, \qquad \overline{\psi} \to \overline{\psi}\,e^{i\theta\gamma_5}\,. \tag{2.40}$$

   Hint: For the second transformation, use a Taylor expansion of $e^{-i\theta\gamma_5}$.

2. The $\psi \to e^{i\theta}\,\psi$ transformation is clearly vectorial, that is, ψ_L and ψ_R transfrom in the same way:

$$\psi \to e^{i\theta}\,\psi = e^{i\theta}\,\psi_L + e^{i\theta}\,\psi_R. \tag{2.41}$$

   The $\psi \to e^{i\theta\gamma_5}\,\psi$ transformation, on the other hand, is chiral. That is, ψ_L and ψ_R transfrom diferently under it:

$$\psi \to e^{i\theta\gamma_5}\,\psi = e^{-i\theta}\,\psi_L + e^{+i\theta}\,\psi_R. \tag{2.42}$$

   Prove Eq. (2.42).

3. Show that the Lagrangian (2.38) is invariant under both transformations of Eq. (2.39).

4. Add a Dirac mass term to the Lagrangian, $m\overline{\psi}\psi$, and show that it breaks the chiral symmetry, $\psi \to e^{i\theta\gamma_5}\,\psi$, and conserves the vectorial one, $\psi \to e^{i\theta}\,\psi$.

The above result is the source of the statement that we can use chiral symmetries to forbid mass terms for fermions.

Question 2.4: Vectorial and chiral symmetries

Consider a system with N Dirac fermion fields $\psi_1, \psi_2, \dots, \psi_N$, and one real scalar field, ϕ. The most general part of the Lagrangian that involves the fermions is

$$\mathcal{L} = \overline{\psi}_i [i\partial\!\!\!/ \delta_{ij} - m_{ij} - \lambda_{ij}\phi]\psi_j \,. \qquad (2.43)$$

Generally, the symmetry of (2.43) is a $U(1)$ under which all the fermions carry the same charge and the scalar carries charge zero.

1. Show that, if $\lambda_{ij} \propto m_{ij}$, the symmetry is larger, $[U(1)]^N$.
2. Find a symmetry to impose on the Lagrangian in Eq. (2.43) that sets $m_{ij} = 0$ but allows for $\lambda_{ij} \neq 0$. Explain why this symmetry must be chiral.

Question 2.5: Accidental symmetries

Consider a variation of the model discussed in Subsection (2.1.4). We impose a $U(1)_{\text{imp}}$ symmetry. The model has two complex scalar fields, ϕ_1 and ϕ_2, that carry charges $q_1^{\text{imp}} = 1$ and $q_2^{\text{imp}} = 5$. The most general Lagrangian for this model is given by Eq. (2.16) that we rewrite here:

$$\mathcal{L} = \partial^\mu \phi_i^\dagger \partial_\mu \phi_i - m_i^2 \phi_i^\dagger \phi_i - \lambda_{ij} (\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j). \tag{2.48}$$

1. Show that $\mathcal{L}$ has a $U(1)_a \times U(1)_b$ symmetry, where the the charges of the fields are

$$q_1 = (1, 0), \qquad q_2 = (0, 1). \tag{2.49}$$

   Here the first [second] number in the parenthesis refers to the charge under $U(1)_a$ [$U(1)_b$].

The $U(1)_a \times U(1)_b$ symmetry is partially accidental, that is, we imposed a single $U(1)$ and ended up with a $[U(1)]^2$ symmetry. We can rewrite the symmetry as a product of $U(1)_{\text{imp}} \times U(1)_{\text{acc}}$ such that $U(1)_{\text{imp}}$ is the imposed symmetry (with $q_1^{\text{imp}} = 1$ and $q_2^{\text{imp}} = 5$) and $U(1)_{\text{acc}}$ is an accidental one. While there are many possible choices for $U(1)_{\text{acc}}$, the one we consider below is particularly useful.

2. We choose $U(1)_{\text{acc}}$ such that the charges under $U(1)_{\text{imp}}$ and $U(1)_{\text{acc}}$ are orthonormal:

$$q_1^{\text{imp}} q_1^{\text{acc}} + q_2^{\text{imp}} q_2^{\text{acc}} = 0, \qquad \left(q_1^{\text{imp}}\right)^2 + \left(q_2^{\text{imp}}\right)^2 = (q_1^{\text{acc}})^2 + (q_2^{\text{acc}})^2 . \tag{2.50}$$

   Find one possibility for q_1^{acc} and q_2^{acc}.

3. A generic $[U(1)]^2$ transformation can be written in the two bases as

$$\phi \to e^{i(q_a \theta_a + q_b \theta_b)} \phi = e^{i(q^{\text{imp}} \theta_{\text{imp}} + q^{\text{acc}} \theta_{\text{acc}})} \phi \tag{2.51}$$

   The angles, θ_{imp} and θ_{acc}, can be expressed in terms of θ_a and θ_b. Find that relation.

4. Write a dimension six operator that breaks the $U(1)_a \times U(1)_b$ symmetry into $U(1)_{\text{imp}}$.