Chapter 6

Spontaneous Symmetry Breaking

Spontaneously broken symmetries play an important role in physics, and in particle physics in particular. In this chapter we introduce the idea of spontaneous symmetry breaking and discuss its consequences. The role of such symmetries in the weak interaction part of the SM is discussed in Chapter 7.

6.1 Introduction

The notion of broken symmetries may seem strange: In what sense is there a difference between the case that we call "a broken symmetry" and the case of not having the symmetry at all? The idea of a broken symmetry is however meaningful in two scenarios:

- Explicit breaking of a symmetry by a small parameter. The Lagrangian includes terms that break the symmetry, but these terms are characterized by a small parameter. The small parameter can be either a small dimensionless coupling, or a small ratio between mass scales. The symmetry is then approximate, and one can expand around the symmetry limit.

- Spontaneous Symmetry Breaking (SSB). The Lagrangian is symmetric, but the vacuum state is not. Even though with SSB the symmetry is not manifest but rather hidden, the number of parameters is the same as in the case of unbroken symmetry. In this sense, the predictive power of a spontaneously broken symmetry is as strong as that of the unbroken symmetry.

SSB is based on the following ingredients. Symmetries of interactions are determined by the symmetries of the Lagrangian. The states, however, do not have to obey these symmetries. Consider, for example, the hydrogen atom. While the Lagrangian is invariant under rotations, an eigenstate does not have to be. Specifically, a state with a finite m quantum number is not invariant under rotation around the z-axis. The fact that there are eigenstates that are not invariant under the symmetry of the system is a generic feature when there are degenerate states.

In perturbative QFT we expand around the lowest energy state. This lowest state is called the "vacuum" state. When the vacuum state is degenerate, the fact that the physics would remain the same for any choice of vacuum to expand around is a consequence of the symmetry. Yet, when we expand around a specific vacuum state out of the degenerate set of vacua, we expand around a state that does not conserve the symmetry.

The name "spontaneously broken" indicates that there is no preference as to which of the states is chosen. A very simple example is that of a hungry donkey. Consider a donkey that stands exactly halfway between two stacks of hay. Symmetry tells us that it costs the same amount of energy to go to either stack. Thus, we may expect that the donkey is unable to choose and would starve! Yet, in reality, the donkey would make an arbitrary choice and go to one of the stacks to eat. We say that the donkey spontaneously breaks the symmetry between the two sides.

In previous chapters we encountered the predictive power of imposed symmetries. In this chapter we show that spontaneously broken symmetries are no less predictive than exact ones, though the predictions are different. While the symmetry is no longer manifest, in the sense that processes that are forbidden in the symmetry limit may become allowed if it is spontaneously broken, there are subtle relations between these 'forbidden' processes and the allowed ones. These relations reveal that the Lagrangian does have this symmetry. This is why a spontaneously broken symmetry is also called a hidden symmetry.

6.2 Global discrete symmetries: Z_2

Consider a model with an imposed Z_2 symmetry, similar to the one discussed in Section 2.1.1. There is a single real scalar field ϕ, which is odd under the symmetry:

$$\phi \to -\phi. \tag{6.1}$$

Thus, the symmetry is simply ϕ-parity. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4. \tag{6.2}$$

In particular, the symmetry forbids a ϕ^3 term. Hermiticity of $\mathcal{L}$ requires that μ^2 and λ are real, and we must have $\lambda > 0$. ($\lambda < 0$ leads to a "run-away" potential, that is, one that is not bounded from below.) As for the μ^2 term, we can have either $\mu^2 > 0$ or $\mu^2 < 0$. The $\mu^2 > 0$ case is considered in Section 2.1.1. It corresponds to an ordinary ϕ^4 theory, and μ^2 is the mass-squared of ϕ. The case of interest for our purposes is

$$\mu^2 < 0. \tag{6.3}$$

The minimum of the scalar potential should satisfy

$$0 = \frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2). \tag{6.4}$$

Thus, the potential has two possible minima:

$$\phi_\pm = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v. \tag{6.5}$$

The classical solution would be either ϕ_+ or ϕ_-. We say that ϕ acquires a Vacuum Expectation Value (VEV):

$$\langle\phi\rangle \equiv \langle 0|\phi|0\rangle \neq 0. \tag{6.6}$$

Perturbative calculations should involve expansion around the classical minimum. Since the two solutions are physically equivalent, the physics cannot depend on our choice, but we must make a choice. Let us choose — without loss of generality — to expand around ϕ_+. We define a field ϕ' with a vanishing VEV:

$$\phi' = \phi - v. \tag{6.7}$$

In terms of ϕ', the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi')(\partial^\mu\phi') - \frac{1}{2}(2\lambda v^2)\phi'^2 - \lambda v\phi'^3 - \frac{\lambda}{4}\phi'^4, \tag{6.8}$$

where we used $\mu^2 = -\lambda v^2$ and discarded a constant term.

We recall that the most general Lagrangian of a scalar field is given in Eq. (1.2) and we rewrite it here in terms of ϕ'

$$\mathcal{L}_S = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{m^2}{2}\phi'^2 - \frac{\eta}{2\sqrt{2}}\phi'^3 - \frac{\lambda}{4}\phi'^4\,. \tag{6.9}$$

Examining the Lagrangians of Eqs. (6.2), (6.8) and (6.9) raises the following points:

1. The Lagrangian (6.8) includes all possible terms for the real scalar field ϕ'. In particular, it has no ϕ'-parity symmetry. Thus, the $\phi \to -\phi$ symmetry is hidden. It is spontaneously broken by our choice of the ground state $\langle\phi\rangle = +v$.

2. Yet, the Lagrangian (6.8) is not the most general renormalizable Lagrangian for a scalar field. While the most general one, Eq. (6.9), depends on three independent parameters, Eq. (6.8) depends on only two. In terms of the parameters of the general Lagrangian Eq. (6.9), the relation is

$$\eta^2 = 4\lambda m^2. \tag{6.10}$$

This relation is the clue that the symmetry is spontaneously, rather than explicitly, broken.

3. The two parameters can be chosen to be v and λ, or μ^2 and λ. The first choice is the one we made in writing Eq. (6.8). The second choice employs the same parameters of the original, manifestly symmetric $\mathcal{L}(\phi)$, see Eq. (6.2). It demonstrates that the SSB does not introduce additional new parameters.

4. The coefficients of the quadratic and trilinear terms in (6.8) are different from those of the quadratic and trilinear terms in Eq. (6.2). In contrast, the coefficients of the quartic terms are the same. This is a general result: as long as we consider only the renormalizable terms, SSB changes dimensionful parameters, but not dimensionless ones.

5. While the symmetry is manifest in Eq. (6.2), the phenomenological interpretation of this model should start from Eq. (6.8). Specifically, the model describes a scalar particle of mass-squared $2\lambda v^2 = -2\mu^2$. This particle is an excitation of the ϕ' field.

6. Hypothetical particles with negative mass-squared are called tachyons and they travel faster than light. The example above shows why tachyons do not appear in QFT. To make a physical interpretation, we have to expand around the minimum, and thus physical particles have positive mass-squared. Fields with negative mass-squared terms are sometimes referred to as tachyonic fields.

7. In non-relativistic quantum mechanics, the analogous case – a particle in a double-well potential – does not have a degenerate vacuum due to tunneling effects. Such tunneling effectively vanishes in QFT.

6.3 Global Abelian continuous symmetries: $U(1)$

Consider a model with an imposed $U(1)$ symmetry, similar to the one discussed in Section 2.1.2. There is a single complex scalar field ϕ, with $q = +1$, so the theory is required to be invariant under

$$\phi \to e^{i\theta}\phi. \tag{6.11}$$

The Lagrangian reads

$$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{6.12}$$

Equivalently, we can rewrite the Lagrangian in terms of two real scalar fields, as in Eq. (2.5),

$$\phi \equiv \frac{1}{\sqrt{2}}\left(\phi_R + i\phi_I\right), \tag{6.13}$$

and impose an $SO(2)$ symmetry,

$$\begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}. \tag{6.14}$$

The Lagrangian reads, see Eq. (2.8),

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi_R)(\partial_\mu\phi_R) + \frac{1}{2}(\partial^\mu\phi_I)(\partial_\mu\phi_I) - \frac{\mu^2}{2}\left(\phi_R^2 + \phi_I^2\right) - \frac{\lambda}{4}\left(\phi_R^2 + \phi_I^2\right)^2. \tag{6.15}$$

The μ^2 and λ parameters are real, and we must have $\lambda > 0$. We consider the case that $\mu^2 < 0$. (The $\mu^2 > 0$ case is considered in Section 2.1.2.) We define $v^2 = -\mu^2/\lambda$. The scalar potential can be written (up to a constant term) as

$$V = \lambda \left(\phi^\dagger \phi - \frac{v^2}{2} \right)^2 . \tag{6.16}$$

Thus, ϕ acquires a VEV:

$$2 \left\langle \phi^\dagger \phi \right\rangle = \left\langle \phi_R^2 + \phi_I^2 \right\rangle = v^2 = -\frac{\mu^2}{\lambda}. \tag{6.17}$$

In the (ϕ_R, ϕ_I) plane, there is a circle of radius v that corresponds to minima of the potential. We have to choose a specific vacuum to expand around. We choose the real component of ϕ to carry the VEV:

$$\langle \phi_R \rangle = v, \qquad \langle \phi_I \rangle = 0. \tag{6.18}$$

We define the real scalar fields

$$h = \phi_R - v, \qquad \xi = \phi_I, \tag{6.19}$$

with vanishing VEVs:

$$\langle h \rangle = \langle \xi \rangle = 0. \tag{6.20}$$

We obtain the Lagrangian in terms of h and ξ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) - \lambda v^2 h^2 - \lambda v h (h^2 + \xi^2) - \frac{\lambda}{4}(h^2 + \xi^2)^2. \tag{6.21}$$

Note the following points:

1. The $SO(2)$ symmetry is spontaneously broken. This can be seen from the presence of the $h(h^2 + \xi^2)$ term.

2. Since the symmetry is spontaneously broken, the Lagrangian is not invariant under a transformation similar to the one in Eq. (6.14). Explicitly,

$$\begin{pmatrix} h \\ \xi \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h \\ \xi \end{pmatrix} \tag{6.22}$$

   is not a symmetry of $\mathcal{L}$.

3. The Lagrangian describes one massive scalar, h, with $m^2 = 2\lambda v^2$, and one massless scalar, ξ. If the symmetry were not broken, it would be impossible to distinguish the two components of the complex scalar field. With the symmetry spontaneously broken, these two DoF are distinguishable, for example, by their different masses.

4. The Lagrangian of Eq. (6.21) is not the most general Lagrangian for two real scalar fields. Many terms are missing, while others, that would have been independent in the general case, are related. In particular, there are only two independent parameters, as for a Lagrangian with an unbroken $SO(2)$.

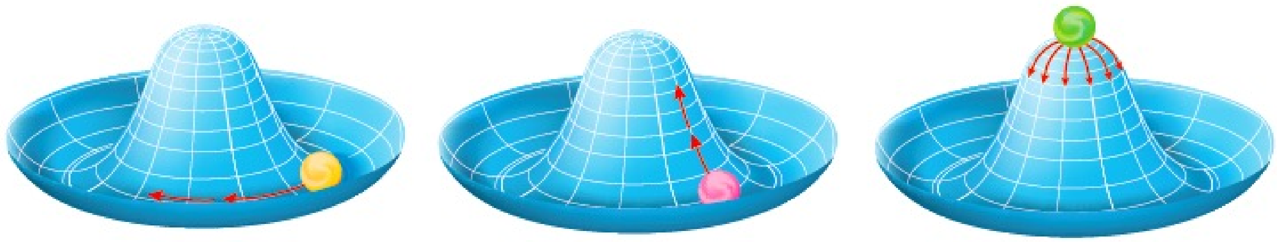

Figure 6.1: The "Mexican hat" potential. The masses of the two scalar DoF correspond to the second derivative of the potential around the minimum. One direction (left) is flat, while the other (center) is not. The plot on the right shows the symmetric maximum. It is unstable and does not correspond to a particle. In the case of a global symmetry, the flat direction corresponds to the massless Goldstone boson, while the non-flat direction corresponds to the massive DoF. In the case of a local symmetry, the flat direction corresponds to the longitudinal component of the vector boson, while the non-flat direction corresponds to the massive Higgs boson.

5. The quartic terms, with dimensionless couplings, are the same in Eqs. (6.15) and (6.21). Only dimensionful couplings are modified.

6. We chose a basis by assigning the VEV to the real component of ϕ. This is an arbitrary choice. We made it since it is convenient. The physics does not depend on this choice.

7. We write the VEV as $\langle\phi_R\rangle = v$ or equivalently as $\langle\phi\rangle = v/\sqrt{2}$. The factor of $\sqrt{2}$ between the two VEVs is just the one that we encounter many times when moving between real and complex fields.

One of the most interesting features of the model presented here is the existence of a massless scalar field. This feature is not particular to our specific model, but rather the result of a general theorem called Goldstone's theorem: The spontaneous breaking of a global continuous symmetry is accompanied by massless scalars. Their number and quantum numbers equal those of the broken generators. The massless scalars are called Nambu-Goldstone Bosons.

While we do not prove here the theorem, we briefly describe the intuition behind it. SSB is possible only when the vacuum is degenerate. For a continuous symmetry, the set of degenerate vacua is also continuous. In the case of a $U(1)$ symmetry, the shape of the potential is usually called "a Mexican hat," see Fig. 6.1. When expanding around any point in the "valley," one direction is flat. A flat direction in the potential corresponds to a massless DoF. Goldstone's theorem is a generalization of this simple picture. Fig. 6.1 demonstrates the point.

6.4 Global non-Abelian continuous symmetries: $SO(3)$

Consider a model with an imposed $SO(3)$ symmetry, and a real scalar field, ϕ, that transforms as a triplet under the symmetry:

$$\phi \to e^{iL_a\theta_a}\phi. \tag{6.23}$$

The three L_a matrices, $(L_a)_{bc} = i\epsilon_{abc}$, constitute the triplet representation of the $SO(3)$ algebra. They are given explicitly in Eq. (6.49). This model provides an intuitive picture of spontaneous symmetry breaking. The triplet constitutes a vector in a real three-dimensional (3d) space. Once a vector is fixed in space, the symmetry under 3d rotations breaks, but not completely, as the symmetry under rotations in the plane that is perpendicular to the vector remains. Here we translate this intuitive picture into a rigorous analysis.

The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^T)(\partial^\mu\phi) - \frac{\mu^2}{2}\phi^T\phi - \frac{\lambda}{4}(\phi^T\phi)^2. \tag{6.24}$$

We take $\mu^2 < 0$, and define $v^2 = -\mu^2/\lambda$. Then ϕ acquires a VEV: $|\langle\phi\rangle| = v$. The triplet ϕ has three DoF. We choose the direction of the VEV to lie in the ϕ_3 direction:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ v + \phi_3 \end{pmatrix}, \tag{6.25}$$

such that $\langle\phi_i\rangle = 0$, $i = 1, 2, 3$. The $SO(3)$ symmetry is only *partially* broken,

$$SO(3) \to SO(2), \tag{6.26}$$

where the $SO(2)$ symmetry refers to rotations in the (ϕ_1, ϕ_2) plane.

The Lagrangian for the ϕ_i fields can be written as

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \tag{6.27}$$

where $\mathcal{L}_n$ includes terms that are n'th power in the ϕ_i fields. Let us comment on the significance of each of these parts of the Lagrangian:

- Quadratic terms:

$$-\mathcal{L}_2 = \lambda v^2\phi_3^2. \tag{6.28}$$

  The model has one massive scalar, ϕ_3, of mass-squared $m_3^2 = 2\lambda v^2$, and two massless scalars, $m_1^2 = m_2^2 = 0$. This is a manifestation of Goldstone's theorem. Spontaneous symmetry breaking requires the appearance of massless Goldstone bosons in correspondence to the broken generators. Since $SO(3)$ has three generators, and it is spontaneously broken to $SO(2)$ that has one generator, our model must have two Goldstone bosons.

- Trilinear terms:

$$-\mathcal{L}_3 = \lambda v \phi_3(\phi_1\phi_1 + \phi_2\phi_2 + \phi_3\phi_3). \tag{6.29}$$

  The fact that $\mathcal{L}_3 \neq 0$ is a manifestation of the $SO(3)$ breaking.

- The quartic terms:

$$-\mathcal{L}_4 = \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)^2. \tag{6.30}$$

  This part of the Lagrangian has dimensionless couplings and therefore it is unchanged from the symmetric form.

The model presented here is an example of partial breaking of the symmetry. In general, a generator corresponds to a spontaneously broken symmetry if the vacuum is not invariant under an operation of the corresponding group element. Conversely, a generator corresponds to an unbroken symmetry if the vacuum is invariant to an operation by the corresponding group element. Given the fact that the group element is the exponent of the generator, these conditions can be represented in terms of the generators as follows. We denote the the vacuum state by $\langle\phi\rangle$. A broken generator gives

$$T_a\langle\phi\rangle \neq 0. \tag{6.31}$$

An unbroken generator gives

$$T_a\langle\phi\rangle = 0. \tag{6.32}$$

More details are given in Question 6.3.

6.5 Fermion masses

Spontaneous symmetry breaking can give masses to chiral fermions. We explain this statement by an explicit example.

Consider a model with a $U(1)$ symmetry. The field content consists of a left-handed fermion ψ_L, a right-handed fermion ψ_R, and a complex scalar ϕ with the following $U(1)$ charges:

$$q(\psi_L) = +1, \qquad q(\psi_R) = +2, \qquad q(\phi) = +1. \tag{6.33}$$

The most general Lagrangian we can write is

$$\mathcal{L} = \mathcal{L}_{\rm kin} - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - (Y\phi\,\overline{\psi_R}\psi_L + \text{h.c.}). \tag{6.34}$$

Since the fermions are charged and chiral, we cannot write mass terms for them ($\mathcal{L}_\psi = 0$).

We take $\mu^2 < 0$, so that the scalar potential is the one given in Eq. (6.15), leading to a VEV for ϕ: $|\langle\phi\rangle| = v/\sqrt{2} \neq 0$. As in Section 6.3, we choose $\langle\phi_R\rangle = v$, $\langle\phi_I\rangle = 0$ and define the real fields h and ξ in such a way that they have vanishing VEVs:

$$\phi = \frac{h + v + i\xi}{\sqrt{2}}, \tag{6.35}$$

Expanding around the chosen vacuum we find

$$\mathcal{L} = \mathcal{L}_{\rm kin} - V(h,\xi) - \left[\frac{Yv}{\sqrt{2}}\overline{\psi_R}\psi_L + \frac{Y}{\sqrt{2}}(h+i\xi)\overline{\psi_R}\psi_L + {\rm h.c.}\right], \tag{6.36}$$

where $V(h,\xi)$ can be read off Eq. (6.21). We learn that ψ_L and ψ_R combine to form a Dirac fermion with mass

$$m_\psi = \frac{Yv}{\sqrt{2}}. \tag{6.37}$$

This is possible because the symmetry under which the fermion is chiral is broken.

In a more general case, the symmetry might be only partially broken, namely a subgroup of the original group remains unbroken. In this case, necessary conditions for generating fermion masses are the following:

- Dirac mass: the fermion representation is vector-like under the unbroken subgroup.
- Majorana mass: the fermion is neutral under unbroken $U(1)$ groups and in a real representation of unbroken non-Abelian subgroups.

6.6 Local symmetries: the Higgs mechanism

In this section we discuss spontaneous breaking of local symmetries. We demonstrate it by studying a $U(1)$ gauge symmetry. One of the main results is that the breaking of a local symmetry generates mass terms for the gauge bosons that correspond to the broken generators. At first sight, this result might seem surprising, since the spontaneous breaking of a global symmetry gives massless Nambu-Goldstone bosons. In the case of a local symmetry, however, these would-be Nambu-Goldstone bosons are "eaten" by the gauge bosons and become the longitudinal components of the resulting massive vector-bosons.

An explanation of the terms "eaten" and "would-be Goldstone bosons", that are commonly used in the physics jargon, is called for. The key point for the term "eaten" is that, for a model with a spontaneously broken local symmetry, the number of DoFs is the same in the interaction basis and in the mass basis. Concretely, if N_H of the symmetry generators are spontaneously broken then, in the mass basis, there are N_H massive vector boson fields, each with 3 DoFs (two transverse and one longitudinal polarizations). In the interaction basis, these $3N_H$ DoFs are assigned to N_H gauge fields, each with 2 DoFs (the two transverse polarizations), and N_H real scalar fields. These N_H real scalar fields of the interaction basis, which become part (the longitudinal components) of the N_H massive vector bosons in the mass basis, are referred to as the "eaten" DoFs. The term "would-be Goldstone bosons" refers to the fact that, if the spontaneously broken symmetry were global instead of local, these N_H real scalar fields of the interaction basis would correspond to the massless Goldstone bosons in the mass basis.

Consider a theory similar to the one discussed in Section 6.3, where we have a single scalar field that is charged under a $U(1)$ symmetry. The difference is that here we impose a local $U(1)$ symmetry:

$$\phi \to e^{i\theta(x)}\phi. \tag{6.38}$$

The Lagrangian is given by

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{6.39}$$

The covariant derivative is given by

$$D^\mu\phi = (\partial^\mu + igA^\mu)\phi\,. \tag{6.40}$$

A^μ is the gauge field, $F_{\mu\nu}$ is defined in Eq. (2.28), and g is the coupling constant.

We consider the case of $\mu^2 < 0$, leading to SSB via a VEV of ϕ:

$$\langle\phi\rangle = \frac{v}{\sqrt{2}}, \qquad v^2 = -\frac{\mu^2}{\lambda}. \tag{6.41}$$

We choose the real component of ϕ to carry the VEV. We again write the complex scalar in terms of two real scalar fields with vanishing VEVs, $\langle h\rangle = \langle\xi\rangle = 0$, but, unlike the global case, it is convenient to write the two DoF as a phase, $\xi(x)$ and a magnitude, $h(x)$:

$$\phi(x) = e^{i\xi(x)/v}\frac{v + h(x)}{\sqrt{2}}. \tag{6.42}$$

Note that we normalized $\xi(x)$ such that it has mass dimension one. To linear order in the fields, Eq. (6.42) is the same as Eq. (6.35). We usually refer to Eq. (6.42) as a non-linear realization and to Eq. (6.35) as a linear realization.

When a symmetry is spontaneously broken and we write the Lagrangian in terms of the VEV-less fields, the Lagrangian is no longer manifestly invariant under the broken symmetry transformation. Instead, the transformation constitutes a change of basis. We can use this change of basis to our advantage, by choosing a basis that makes the physics of the model more transparent. This is what we do here by choosing a specific gauge: $\theta(x) = -\xi(x)/v$. (It is fully legitimate to choose the phase to be related to a field.) This gauge is called the unitary gauge.

With this choice of gauge,

$$\phi \to \phi' = \frac{1}{\sqrt{2}}(h + v), \qquad A_\mu \to V_\mu = A_\mu + \frac{1}{gv}\partial_\mu\xi, \tag{6.43}$$

such that ϕ' has one DoF and V_μ three. The Lagrangian in terms of h and V_μ reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}(g^2v^2)V_\mu V^\mu - \frac{1}{2}(2\lambda v^2)h^2 \\ &+\frac{g^2}{2}V_\mu V^\mu h(2v + h) - \lambda v h^3 - \frac{\lambda}{4}h^4.\end{aligned} \tag{6.44}$$

The kinetic term of the gauge boson is independent of the gauge fixing. This can be seen from the fact that

$$\partial_\mu V_\nu - \partial_\nu V_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.45}$$

The spectrum of the model consists of a massive vector boson of mass-squared $m_V^2 = (gv)^2$ and a massive scalar of mass-squared $m_h^2 = 2\lambda v^2$. We note the following points:

1. The sign of a mass-squared term for a vector-boson is opposite sign to that of a mass-squared term for a scalar.

2. The h scalar is called "a Higgs boson". The related field, which acquires a VEV, in our case ϕ, is called the Brout-Englert-Higgs (BEH) field or the Higgs field.

3. The source of the mass-squared term for the vector bosons is the kinetic term of the Higgs field.

4. The propagator of a a massive gauge boson depends on the gauge choice. In the unitary gauge it is given by

$$(-i)\frac{g^{\mu\nu} - (k^\mu k^\nu)/m_V^2}{k^2 - m_V^2}. \tag{6.46}$$

   We do not discuss in detail the issues of gauge fixing for massive gauge bosons.

5. The ξ field is "eaten" in order to give mass to the gauge boson. It was a convenient choice to make the phase to be the "eaten" DoF. The total number of degrees of freedom does not change: instead of the scalar ξ, we have the longitudinal component of a massive vector boson.

6. In the limit $g \to 0$ we have $m_V \to 0$. This situation describes a massless gauge boson and a massless scalar. We see that in that limit the longitudinal component is the massless Nambu-Goldstone boson as expected.

The interactions of the model include scalar self-interactions and interactions of the scalar with the vector boson. We note the following points:

1. The hVV coupling is proportional to the mass-squared of the vector boson.

2. The dimensionless $VVhh$ and $hhhh$ couplings are unchanged from the symmetric Lagrangian.

The Lagrangian (6.39) depends on three parameters. They can be taken to be g, v, and λ. The Lagrangian (6.44) has two mass terms and four interaction terms which depend on the same three parameters. Thus, the six relevant terms, which would be independent in the absence of a symmetry, obey three relations among them. This is a sign of SSB.

In the example above, we consider SSB of a local $U(1)$ symmetry. The basic ingredients are, however, much more generic and apply also to non-Abelian symmetries and to product groups.

Table 6.1: Symmetries and some of their main consequences

Type	Consequences
Spacetime	Conservation of energy, momentum, angular momentum
Discrete	Selection rules
Global (exact)	Conserved charges
Global (spon. broken)	Massless scalars
Local (exact)	Interactions, massless spin-1 mediators
Local (spon. broken)	Interactions, massive spin-1 mediators

In fact, the SM incorporates SSB of a local $SU(2) \times U(1)$ symmetry. The following lessons are generic to all cases of spontaneous breaking of a local symmetry:

- Spontaneous symmetry breaking gives masses to the gauge bosons related to the broken generators.
- Gauge bosons related to an unbroken subgroup remain massless, because their masslessness is protected by the symmetry.

The following points are common to the spontaneous breaking of both local and global symmetries:

- The field that acquires a VEV (the BEH field) must be a scalar field. Otherwise its VEV would break Lorentz invariance.
- Spontaneous breaking of a symmetry, whether global or local, can give masses also to fermions, via Yukawa interactions.
- States with different QNs under the broken symmetry but with the same QNs under the unbroken subgroup can mix. By "mixing" we mean that a mass eigenstate can be a linear combination of such states. We do not elaborate on it here. Chapter 7 provides an example of mixing among vector bosons. Chapter 14 provides an example of mixing among fermions.

6.7 Summary

Symmetries in QFT have a strong predictive, or explanatory, power. The main consequences of the various types of symmetries are summarized in Table 6.1.

To construct a model, we provide as input the following ingredients:

(i) The symmetry;

(ii) The transformation properties of the fermions and the scalars.

Then we write the most general Lagrangian that is invariant under the symmetry up to some order in the fields. Unless explicitly stated otherwise, we truncate the Lagrangian at the renormalizable level, that is, at dimension four in the fields.

The resulting Lagrangian has a finite number of parameters that we need to determine by experiment. In principle, for a theory with N independent parameters, we need to perform N appropriate measurements to extract the values of the parameters. Additional measurements test the theory.

The values of the parameters can have minor or major implications. In particular, variation of values of parameters can lead to different patterns of SSB, which result in very different phenomenology. Thus, often, when we define a theory, the various SSB branches of it are given different names.

Our process of building some model X starts with defining the imposed symmetry and the transformation properties of fermions and scalars under this symmetry, and writing the most general Lagrangian consistent with these definitions. At this stage, we can obtain the predictions of model X that are independent of the values of the model parameters. Additional predictions can be made when the values of the parameters are determined experimentally. We sometimes use the term "*a* model X" for the *class* of models that are described by the Lagrangian, with any possible values of its parameters, and "*the* model X" for the *specific* model with the values of the parameters as realized in Nature (namely, as determined by experiments). In particular, we use the terminology of "a SM" and "the SM" in this sense later in the book.

For further reading

More on the formal aspects of SSB can be found, for example, in Section 2.2 in Ref. [17], Chapters 20 and 21 of Ref. [2], Chapter 28 in Ref. [15], and Section 4 of Ref. [21].

Problems

Question 6.1: Algebra

1. Show that, to up to linear order in all fields, Eq. (6.42) leads to Eq. (6.35).
2. Use Eqs. (6.39), (6.42), and (6.43) to derive Eq. (6.44).

Question 6.2: SSB with many scalars

Consider the following model. The symmetry is global $SO(N)$. There is a real scalar field Φ, in the N representation, so there are N real scalar DoF $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\mu^2 \Phi^2 - \frac{1}{4}\lambda \Phi^4, \tag{6.47}$$

with $\mu^2 < 0$ and $\lambda > 0$. This Lagrangian is a generalization of Eq. (6.12).

1. Show that $\mathcal{L}$ describes a theory with a single massive scalar of mass-squared $m^2 = -2\mu^2$, and $N - 1$ massless scalars.
2. What is the unbroken symmetry group?
3. Goldstone's theorem states that the number of massless bosons is equal to the number of broken generators. Show this explicitly for this model.

Question 6.3: Broken and unbroken symmetries

In this question we elaborate on Eqs. (6.31) and (6.32) that state that an unbroken generator T_a annihilates the vacuum, $T_a\langle\phi\rangle = 0$, while a spontaneously broken one does not, $T_a\langle\phi\rangle \neq 0$. Consider the operation of a group element on the vacuum:

$$\langle\phi'\rangle = e^{iT_a\theta_a}\langle\phi\rangle. \tag{6.48}$$

1. Explain why the symmetry is unbroken if $\langle\phi'\rangle = \langle\phi\rangle$ for any θ_a and that it is broken if there is a θ_a such that $\langle\phi'\rangle \neq \langle\phi\rangle$.

2. Explain why the above implies that $T_a\langle\phi\rangle = 0$ if T_a corresponds to an unbroken symmetry.

3. A familiar example is the case of a vector in 3d, that breaks the symmetry from $SO(3)$ to $SO(2)$, that is, from rotations in 3d to rotations in the plane perpendicular to the vector. Consider a case where we choose the normalized vector to be $\vec{v} = (0,0,1)^T$. Show that L_z is still a symmetry while L_x and L_y are not. It is useful to recall the explicit representation of L_i for a vector, in the basis that corresponds to rotations in real space:

$$L_x = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad L_y = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad L_z = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.49}$$

4. Now consider a generic normalized vector, $\vec{v} = (a,b,c)$ such that $a^2+b^2+c^2 = 1$. Show that

$$[aL_x + bL_y + cL_z]\,\vec{v} = 0. \tag{6.50}$$

   The above shows that there is always one generator that is not broken, so indeed the unbroken symmetry is $SO(2)$.

5. Consider $SU(2)$ transformations. The previous question demonstrates that a vector, that is a **3**, breaks $SU(2)$ to $U(1)$. Here you are asked to show that a spinor, that is a **2**, breaks $SU(2)$ completely. In order to show this, we need to prove that there is no combination of generators which annihilates a spinor. The $SU(2)$ generators in the spinor representation are the Pauli matrices. Consider the spinor $\vec{s} = (0,1)$ and show that any non-zero linear combination with real coefficients of the Pauli matrices does not annihilate it.

Question 6.4: More on the dark photon

We consider a model that is an extension of the one discussed in Question 3.3:

(i) The symmetry is a local $U(1)_{\rm EM} \times U(1)_D$. We denote the gauge bosons by A_μ and C_μ, respectively.

(ii) There are four fermion fields:

$$e_L(-1,0), \qquad e_R(-1,0), \qquad d_L(0,-1), \qquad d_R(0,-1). \tag{6.51}$$

(iii) There is a single complex scalar:

$$\phi(q_{\rm EM}, q_D). \tag{6.52}$$

We assume no kinetic mixing and use a normalization such that the coupling constants of the two groups is the same, that is, $g_{\rm EM} = g_D = e$.

1. There are five specific charge assignments that allow Yukawa interactions, that is, couplings between ϕ and the fermions. What are these charge assignments?

From this point on, we do not consider any of the above options, that is, we consider only cases where all Yukawa interactions are forbidden.

2. Write the scalar potential. What is the condition for ϕ to acquire a VEV? From here on, assume that this condition is satisfied.

3. One way to make the model possibly consistent with Nature is to have partial SSB, such that the photon A_μ is massless but the dark photon C_μ is massive. Explain why this is the case when $q_{\rm EM} = 0$ and $q_D \neq 0$.

4. In the above case, that is with $q_{\rm EM} = 0$ and $q_D \neq 0$, write the mass of the dark photon in terms of the model parameters.

5. We now consider a case where both $q_{\rm EM} \neq 0$ and $q_D \neq 0$. In this case both $U(1)_{\rm EM}$ and $U(1)_D$ are broken. Show, however, that the breaking pattern is $[U(1)]^2 \to U(1)$. We denote the massless gauge boson A'_μ and the massive one C'_μ.

6. Write the couplings of the fermions to A'_μ and C'_μ.

7. We now assume that $q_{\rm EM} \ll q_D$ (and $g_{\rm EM} = g_D$). In this case, we can think of A'_μ as a small deviation from A_μ, and still call it the photon. We further assume that $m_d \sim m_e$. Experimentally, a particle with a mass of order the electron mass and with EM charge larger than about 10^{-3} that of the electron, is ruled out. Obtain the resulting constraint on $q_{\rm EM}/q_D$.

Question 6.5: Physics of the Higgs boson

We consider the model of Section 6.3 with the Lagrangian of Eq. (6.21).

1. Draw the tree-level diagrams for the $hh \to hh$ scattering and write down the amplitude. Note that there is more than one diagram.

2. Estimate the cross section in the limit where $E \gg v$. Here E is the center of mass energy of the collision.

3. Consider the same model but with $\mu^2 > 0$. Estimate the $\phi\phi^* \to \phi\phi^*$ cross section in the limit where $E^2 \gg \mu^2$. Explain the similarity to the result of the $hh \to hh$ scattering cross section obtained in the previous item.

Question 6.6: The Sigma model

A classic example of spontaneous symmetry breaking with Nambu–Goldstone bosons is provided by the σ-model. It pre-dated QCD, and we now think of it as an effective theory of the strong interactions at low energies. It aims to describe the effective strong interactions between the nucleons — the proton p and the neutron n — via the exchange of three scalars — the pions π^a $(a = 1, 2, 3)$.

Consider the following model:

(i) The symmetry is a global $SU(2)_L \times SU(2)_R \times U(1)_B$.

(ii) There are two fermion fields:

$$N_L(2,1)_{+1} = \begin{pmatrix} p_L \\ n_L \end{pmatrix}, \qquad N_R(1,2)_{+1} = \begin{pmatrix} p_R \\ n_R \end{pmatrix}. \tag{6.53}$$

(iii) There is a single scalar field:

$$\Sigma(2,2)_0. \tag{6.54}$$

The most general Lagrangian can be written as

$$\mathcal{L} = i\overline{N_L}\partial\!\!\!/ N_L + i\overline{N_R}\partial\!\!\!/ N_R + \frac{1}{4}\mathrm{Tr}\left[\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right] - \left[g(\overline{N}_L\Sigma N_R + \text{h.c.})\right] - V(\Sigma)\,. \tag{6.55}$$

The infinitesimal symmetry transformations on the fermion fields are chiral and given by

$$\delta N_L = i\epsilon_L^a T^a N_L, \qquad \delta N_R = i\epsilon_R^a T^a N_R. \tag{6.56}$$

1. $\mathcal{L}$ is invariant under the above chiral symmetries. The fermion kinetic terms can be written in terms of the Dirac field $N = (N_L\ N_R)^T$. Write the infinitesimal symmetry transformations in the form

$$\delta N = i\epsilon^a T^a N, \qquad \delta N = i\gamma_5\epsilon_5^a T^a N, \tag{6.57}$$

and express ϵ^a and ϵ_5^a in term of ϵ_L^a and ϵ_R^a.

What you showed is that we can write the symmetry in a different basis. Instead of $SU(2)_L \times SU(2)_R$ we can write $SU(2)_V \times SU(2)_A$. The $SU(2)_V$ group is also called the "diagonal $SU(2)$" or "isospin symmetry", while $SU(2)_A$ is usually called the "axial $SU(2)$".

2. Show that a mass term $m\overline{N}N$ is invariant under $SU(2)_V$ but not under $SU(2)_A$. We learn that it is the axial symmetry that forbids fermion masses.

The Σ field has four DoF, so we can write it in terms of four real scalar fields, σ and π^a ($a = 1, 2, 3$):

$$\Sigma = \sigma + i\tau_a\pi^a, \qquad a = 1, 2, 3, \tag{6.58}$$

where τ_a are the Pauli matrices. We aim to have a model that describes Nature, so we need to provide masses to the proton and the neutron. This is done via spontaneous breaking of the chiral symmetry by the Σ field acquiring a VEV.

3. The scalar potential, $V(\Sigma)$, can be written as

$$V(\Sigma) = \frac{\lambda}{4}(\Sigma^\dagger\Sigma)^2 - \frac{m^2}{2}(\Sigma^\dagger\Sigma). \tag{6.59}$$

What are the conditions for $V(\Sigma)$ to be bounded from below and for Σ to acquire a VEV?

4. Show that, up to a constant term, the potential in Eq. (6.59) is equivalent to

$$V(\Sigma) = \frac{1}{4}\lambda\left[\sigma^2 + \vec{\pi}^2 - F_\pi^2\right]^2, \tag{6.60}$$

with λ and F_π real and positive. F_π is the so called "pion decay constant" and it is the only mass scale in the theory.

5. What are the minima of $V(\Sigma)$? Find a minimum where only σ acquires a VEV, but the π_a's do not.

6. Rewrite $\mathcal{L}$ in terms of fields that do not carry a VEV, that is, N_L, N_R, π_a, and $s \equiv \sigma - F_\pi$. What are the masses of these fields? How many DoF are massless?

7. How many generators are broken? Check your result against Goldstone's theorem, that is, check that the number of massless scalars is the same as the number of broken generators.

8. We denote the $\overline{N}N\pi$ interaction coupling by $g_{\pi NN}$. Show that the following relation between masses and couplings holds:

$$m_N = g_{\pi NN}F_\pi. \tag{6.61}$$

This relation is known as the Goldberger-Treiman relation. It is satisfied in Nature to a good accuracy. Such a relation between masses and couplings is a signal of SSB, as discussed in the main text.

9. In Nature the pions have small masses (compared to the nucleon), which reflect a small explicit breaking of a symmetry. What is this broken symmetry: $SU(2)_V$ or $SU(2)_A$?

10. In Nature there is a very small mass splitting between the proton and the neutron, while the model predicts that they are degenerate. Thus the model provides a very good approximation to Nature. What symmetry has to be broken to generate the splitting, $SU(2)_V$ or $SU(2)_A$?

We conclude that the Sigma model predicts that the proton and neutron are degenerate and that the pions are massless. These two predictions are approximately fulfilled in Nature. The model also predicts the existence of the s particle, which can be identified as the $f_0(500)$ resonance. We discuss low energy QCD in more detail in Chapter 10.