Chapter 4

Non-Abelian symmetries

In previous chapters, we discussed Abelian symmetries, such as $U(1)$. For this type of symmetries, also known as commutative symmetries, the result of applying two symmetry transformations does not depend on the order in which they are applied. In this chapter, we discuss non-Abelian symmetries, such as $SU(2)$ and $SU(3)$. For this type of symmetries, also known as non-commutative symmetries, the result of applying two symmetry transformations might depend on the order of which they are applied. Within the SM, and in models that extend it, both Abelian and non-Abelian symmetries play an important role.

4.1 Introduction

In order to discuss non-Abelian symmetries, we use the language of Lie groups (we provide a short review of the topic in Appendix A). For the sake of concreteness, we focus on $SU(N)$ groups, but most of the results are generic and apply to any Lie group.

We are interested in internal symmetries where the symmetry transformation is unitary:

$$\phi \to U\phi, \qquad UU^\dagger = U^\dagger U = \mathbf{1}\,. \tag{4.1}$$

We now emphasize some of the differences between transformation laws for Abelian and non-Abelian symmetries. Specifically, we list differences between $U(1)$ and $SU(N)$ symmetries.

Consider a $U(1)$ symmetry and a field ϕ that carries charge $q \neq 0$.

- The field ϕ is necessarily complex.
- The transformation law for this field is

$$\phi \to e^{iq\theta}\phi. \tag{4.2}$$

- The charge, q, is a real number.

- Thus, for an Abelian symmetry, the transformation law is defined for each single complex field separately. A $U(1)$ symmetry operation changes the phase of the field proportionally to the charge.

- For a Lagrangian to be invariant under a $U(1)$ symmetry, each term must consist of a product of fields such that the sum of their charges is zero.

Consider an $SU(N)$ symmetry and a field ϕ in a representation R of dimension $M > 1$. Here ϕ is a vector with M components, ϕ_i with $i = 1, \ldots, M$.

- If R is a complex representation, ϕ is complex. If R is a real representation, ϕ can be real.

- The transformation law for this field is

$$\phi_i \to \left(e^{iT_a\theta_a}\right)_{ij} \phi_j. \tag{4.3}$$

  Here $i, j = 1, \ldots, M$ and $a = 1, \ldots, (N^2 - 1)$. The T_a's are the generators of the $SU(N)$ algebra:

$$[T_a, T_b] = if_{abc}T_c. \tag{4.4}$$

  For the field ϕ in the M-dimensional representation R, the T_a's are represented by $M \times M$ matrices.

- Thus, for a non-Abelian symmetry, the transformation law is defined for each multiplet of fields separately. The $SU(N)$ symmetry operations consist of rotations among the various components within each multiplet.

- For a Lagrangian to be invariant under a non-Abelian symmetry, each term must consist of a product of fields such that the various representations are contracted into a singlet of the symmetry group.

If the symmetry group is not simple, we can consider an independent rotation within each simple subgroup. Then, it is convenient to represent the field as a vector under each of the simple Lie subgroups. When a field undergoes a transformation under one simple subgroup, it is not affected by the other subgroups. For example, consider an $SU(3) \times SU(2)$ group, and a field that is a triplet under $SU(3)$ and a doublet under $SU(2)$. We can denote the field by $\phi_{\alpha i}$, where $\alpha = 1, 2, 3$ is the $SU(3)$-triplet index, and $i = 1, 2$ is the $SU(2)$-doublet index. We can write separately the transformation laws under an $SU(3)$ symmetry transformation and under an $SU(2)$ symmetry transformation:

$$\phi_{\alpha i} \to \left(e^{(i/2)\lambda_a\theta_a}\right)_{\alpha\beta} \phi_{\beta i}, \qquad \phi_{\alpha i} \to \left(e^{(i/2)\tau_b\theta_b}\right)_{ij} \phi_{\alpha j}. \tag{4.5}$$

Here λ_a are the eight 3×3 Gell-Mann matrices, so that $\lambda_a/2$ are the $SU(3)$ generators in the triplet representation, and τ_b are the three 2×2 Pauli matrices, so that $\tau_b/2$ are the $SU(2)$ generators in the doublet representation.

The notation that we use is best explained by examples. For the case of an $SU(3) \times SU(2)$ symmetry, we write the representation of a field ϕ that is an octet under $SU(3)$ and a singlet under $SU(2)$ as $\phi(8,1)$: The first number in parenthesis is the representation under $SU(3)$, and the second one under $SU(2)$. When we consider product groups that include both non-Abelian and Abelian factors, the charges under $U(1)$ symmetries are written as sub-indices. For example, if the symmetry is $SU(3) \times U(1)$, a field that is a triplet under $SU(3)$ and has charge $+1$ under $U(1)$ is written as $\phi(3)_{+1}$. As a final example we take the symmetry to be $SU(3) \times SU(2) \times U(1)$, with a field ϕ that is an $SU(3)$-triplet, an $SU(2)$-doublet and carries $U(1)$ charge of $+1/6$: $\phi(3,2)_{+1/6}$.

4.2 Global symmetries

As described above, in each Lagrangian term, the product of the various representations must be contracted into a singlet of the imposed symmetry. The way we combine representations is presented in Appendix A.8. In most of the upcoming sections, we only make sure that the product of representations contains a singlet, and keep the group indices of the various fields implicit. (We can do so because, in all cases discussed in this book, there is a single way to combine the product of representations into a singlet. In other cases, where there is more than one way to do so, it is important to write the contraction explicitly.)

In this Section, we present three models:

1. A model that demonstrates that mass-squared terms for scalars cannot be forbidden.

2. A model that demonstrates that fermions in vectorial representations have Dirac masses.

3. A model that demonstrates that fermions in chiral and complex representations are massless.

4.2.1 Scalars and $SO(N)$

Consider the following model:

(i) The symmetry is a global

$$SO(N), \tag{4.6}$$

with $N \geq 2$.

(ii) There is a single scalar field in the fundamental representation,

$$\phi(N). \tag{4.7}$$

(*iii*) There are no fermions.

In this model there are N scalar DoF, but we combine them into an N representation and thus we refer to it as a single scalar field.

The most general renormalizable Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2 \,. \tag{4.8}$$

What we mean by $\phi^\dagger\phi$ is the contraction of an $(\overline{N}) \times (N)$ into a singlet of $SO(N)$. Explicitly, the contraction of the $SO(N)$ indices is given by $\phi_i^*\phi_i$.

We note the following points:

1. There is no internal symmetry that one can impose to set $m^2 = 0$ or $\lambda = 0$.

2. In the case of $N = 2$ the symmetry is Abelian, and the model is the same as the one discussed in Section 2.1.2.

4.2.2 Vectorial fermions and $U(N)$

Consider the following model of vectorial fermions:

(*i*) The symmetry is a global

$$U(N) = SU(N) \times U(1), \tag{4.9}$$

with $N \geq 2$.

(*ii*) There are two fermion fields in the fundamental representation:

$$\psi_L(N)_{+1}, \qquad \psi_R(N)_{+1}. \tag{4.10}$$

(*iii*) There are no scalars.

Here, similar to the model of Section 4.2.1, many DoF are combined into an irreducible representation. Given that each Weyl fermion has two DoF, the model has a total of $4N$ DoF. The most general renormalizable Lagrangian is given by

$$\mathcal{L} = i\overline{\psi_L}\,\partial\!\!\!/\,\psi_L + i\overline{\psi_R}\,\partial\!\!\!/\,\psi_R - (m\overline{\psi_L}\psi_R + \text{h.c.}), \tag{4.11}$$

where the group indices are implicit.

We note the following points:

- We cannot write Majorana mass terms since the fermions are charged under $U(1)$.

- Since this model is vectorial, that is, the left-handed ψ_L and right-handed ψ_R transform in the same way, we can combine them into a Dirac fermion ψ and rewrite Eq. (4.11) as follows:

$$\mathcal{L} = i\overline{\psi}\,\partial\!\!\!/\,\psi - m\overline{\psi}\psi. \tag{4.12}$$

4.2.3 Chiral fermions and $U(N) \times U(N)$

Consider the following model of chiral fermions:

(*i*) The symmetry is a global

$$U(N)_L \times U(N)_R = SU(N)_L \times SU(N)_R \times U(1)_L \times U(1)_R, \tag{4.13}$$

with $N \geq 2$.

(*ii*) There are two fermion fields:

$$\psi_L(N,1)_{1,0}, \qquad \psi_R(1,N)_{0,1}. \tag{4.14}$$

(*iii*) There are no scalars.

The most general renormalizable Lagrangian has only kinetic terms:

$$\mathcal{L} = i\overline{\psi_L}\,\partial\!\!\!/\,\psi_L + i\overline{\psi_R}\,\partial\!\!\!/\,\psi_R. \tag{4.15}$$

A few comments are in order:

- As discussed in Section 2.1.5 and demonstrated here, fermion mass terms can be forbidden by a symmetry. This stands in contrast to scalar mass-squared terms which cannot be forbidden by a symmetry, as demonstrated in the model of Section 4.2.1.
- The reason that the mass term vanishes in the Lagrangian of Eq. (4.15) is that the fermion fields of the model are in a chiral representation of the symmetry. This stands in contrast to the model of Section 4.2.2, where the fermion fields are in a vectorial representation of the symmetry, and therefore the Lagrangian of Eq. (4.11) includes a Dirac mass term.
- The $U(N)$ symmetry of Eq. (4.9) is a subgroup of the $U(N)_L \times U(N)_R$ symmetry of Eq. (4.13). It is often called "the vectorial" subgroup.

4.3 Local symmetries

Abelian local symmetries were discussed in Section 2.2. Here we discuss non-Abelian local symmetries, commonly called "Yang-Mills theories." The following properties apply for both Abelian and non-Abelian local symmetries:

- Terms that depend on scalar and/or fermion fields only, but not on their derivatives, and which are invariant under a global symmetry, are also invariant under the corresponding local symmetry.

- The kinetic terms are not invariant under the local symmetry.

From here on, we consider specifically the case of $SU(N)$, but the main points apply to all Lie groups. To achieve invariance under a local non-Abelian symmetry, we must add gauge fields, that we denote by G_a^μ, and replace the derivative $\partial^\mu\phi$ with a covariant derivative $D^\mu\phi$, such that $D^\mu\phi$ and ϕ transform in the same way:

$$\phi \to e^{iT_a\theta_a(x)}\phi, \qquad D^\mu\phi \to e^{iT_a\theta_a(x)}D^\mu\phi. \tag{4.16}$$

The T_a's are the N^2-1 generators of the $SU(N)$ algebra and, for ϕ in an M-dimensional representation, they are represented by $M \times M$ matrices.

The gauge fields that we introduce must restore the local symmetry for N^2-1 independent rotations. Therefore, they themselves must carry an index $a = 1, 2, \ldots, N^2-1$, in contrast to the single gauge field needed for an Abelian local symmetry. Furthermore, the fields in an $SU(N)$-symmetric theory must transform in a well-defined way under the symmetry. It is suggestive then that the gauge fields G_a^μ are in the adjoint representation of $SU(N)$. You are asked to prove that this is indeed the case in Question 4.6. We conclude that the covariant derivative is given by

$$D^\mu = \partial^\mu + igT_aG_a^\mu\,, \tag{4.17}$$

where g is the dimensionless positive coupling constant. The transformation law for G_a^μ is given by (see Question 4.6)

$$G_a^\mu \to G_a^\mu - f_{abc}\theta_b G_c^\mu - \frac{1}{g}\partial^\mu\theta_a. \tag{4.18}$$

The fact that the non-Abelian gauge field is in the adjoint representation of the gauge group and, in particular, that — unlike the Abelian case — it is not a singlet, has significant consequences: It leads to self-interactions of the gauge fields, as we discuss below.

To promote G_a^μ to a dynamical field, we must introduce a kinetic term. Similar to our treatment of the $U(1)$ case, Eq. (2.27), we now define the field strength $G_a^{\mu\nu}$ as

$$[D^\mu, D^\nu] = igT_aG_a^{\mu\nu}. \tag{4.19}$$

Inserting Eq. (4.17) into Eq. (4.19), and using the $SU(N)$ algebra, we obtain

$$T_aG_a^{\mu\nu} = T_a(\partial^\mu G_a^\nu - \partial^\nu G_a^\mu - gf_{abc}G_b^\mu G_c^\nu), \tag{4.20}$$

Further using $\mathrm{tr}(T_aT_b) \propto \delta_{ab}$, we can rewrite Eq. (4.20) as follows:

$$G_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - gf_{abc}G_b^\mu G_c^\nu\,. \tag{4.21}$$

Compared to the Abelian case, Eq. (2.28), the non-Abelian case, Eq. (4.21), has an extra term. This term is the source of self-interactions of the gauge fields.

The kinetic term of non-Abelian gauge fields is given by

$$\mathcal{L}_V = -\frac{1}{4} G_a^{\mu\nu} G_{a\mu\nu}. \tag{4.22}$$

Note the following points with regard to Eq. (4.22):

- Given the kinetic term, G_a^μ is a dynamical field and its excitations are physical particles. For example, as we show later, the gluon is associated with such excitations.
- We work in the canonically normalized basis where the coefficient of the kinetic term is 1/4.
- While a kinetic term is invariant under the gauge transformation, a mass-squared term — $\frac{1}{2}m^2 G_a^\mu G_{a\mu}$ — is not. Local invariance under non-Abelian symmetry implies massless gauge fields in the adjoint representation.
- A total derivative term of the form $G_a^{\mu\nu}\tilde{G}_{a\mu\nu}$ (with $\tilde{G}_{a\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} G_a^{\rho\sigma}$) is Lorentz invariant, gauge invariant and dimension-four. For the Abelian case, we did not include such a term in Eq. (2.29) because it is not physical. For the non-Abelian case, we still do not include it in Eq. (4.22) because it is not physical at the classical level. Yet, at the quantum level, it is physical. We briefly present the physical implications within the SM of such a term for a local $SU(3)$ symmetry in Section 8.A.1.

The non-Abelian gauge fields couple to themselves. This can be seen by replacing $G_a^{\mu\nu}$ in Eq. (4.22) with the explicit expression in Eq. (4.21), which leads to interaction terms that are trilinear and quartic in the gauge fields:

$$\mathcal{L}_{\rm self-interactions} = g f_{abc} (\partial_\mu G_{a\nu}) G_b^\mu G_c^\nu - \frac{1}{4} g^2 (f_{abc} G_b^\mu G_c^\nu)(f_{ade} G_{d\mu} G_{e\nu}). \tag{4.23}$$

The self-interactions of the non-Abelian gauge fields stand in contrast to the Abelian gauge fields which have no self-interactions. As mentioned above, the source of this difference is the fact that $U(1)$ gauge fields are neutral under the $U(1)$, while $SU(N)$ gauge fields are in the adjoint representation of the $SU(N)$, and thus are charged under their own group.

If the symmetry group decomposes into several commuting factors, each factor has its own gauge fields in the corresponding adjoint representation and an independent coupling constant. For example, if the symmetry is $SU(3) \times SU(2) \times U(1)$, we must introduce three irreducible representations of gauge fields: $(8,1)_0$ representation, with coupling constant g_3, to make the Lagrangian invariant under the local $SU(3)$; $(1,3)_0$, with coupling constant g_2, to achieve invariance under local $SU(2)$; and $(1,1)_0$, with coupling constant g_1, to achieve invariance under local $U(1)$.

4.4 Running coupling constants

In QFT, the coupling constants depend on the energy scale. In the Physics jargon, we say that the coupling constants run. Here we do not discuss the theoretical background for this effect and assume that the reader is familiar with it from a QFT course. We just present the relevant equations and mention several important consequences.

The running — namely, the energy dependence — of a coupling g is given by the beta function:

$$\frac{\partial g}{\partial \log \mu} = \beta(g)\,, \tag{4.24}$$

where μ is the relevant energy scale. The beta function depends on the field content of the theory. The leading order effects depend only on fields that are charged under the symmetry. The fact that an Abelian gauge boson is neutral under the Abelian symmetry, while a non-Abelian gauge field is charged under the corresponding non-Abelian symmetry, has important implications on the running of the respective coupling constants.

Consider a local $U(1)$ theory, with n_f Weyl fermion fields with charge $|q| = 1$. The beta function for the coupling constant g_1 is given, to leading order, by

$$\beta(g_1) = \frac{n_f g_1^3}{24\pi^2}\,. \tag{4.25}$$

(This result is often quoted for the case of n_f Dirac fermions, where the factor of 1/24 is replaced by 1/12.)

Consider a local $SU(N)$ theory, with n_f fermion fields in the fundamental representation N and n_f fermion fields in the anti-fundamental representation $\overline{N}$. The beta function for the coupling constant g_N is given, to leading order, by

$$\beta(g_N) = \left(\frac{2n_f - 11N}{3}\right)\frac{g_N^3}{16\pi^2}. \tag{4.26}$$

The important difference between the running effects of the Abelian case, Eq. (4.25), and non-Abelian case, Eq. (4.26), is in the sign of the beta function. In the $U(1)$ case, the beta function is always positive. Consequently, the lower the relevant energy scale, the smaller the coupling. In the $SU(N)$ case, the beta function can assume either sign. If the number of fermions is not too large, $n_f < (11N/2)$ (and in particular in the pure gauge case, $n_f = 0$), the sign is negative and, consequently, the lower the relevant energy scale, the larger the coupling. The phenomenon that couplings become smaller at higher energies goes under the name of "asymptotic freedom."

If a coupling grows large, one loses the ability to make accurate predictions using perturbation theory with the coupling as the small parameter. For $U(1)$ theories, we can use perturbative calculations to make accurate predictions at low enough energy (often called "the IR"). For $SU(N)$ theories, if the number of charged fermions is small, $n_f < 11N/2$, we can use perturbative calculations to make accurate predictions only at high enough energies ("the UV"), while in the IR perturbativity is lost. We discuss the implications of this situation in the next chapter where we present QCD — the theory of strong interactions — as a specific and important example.

Table 4.1: Abelian vs Non-Abelian symmetries

	Abelian	Non-Abelian
Field	Φ_q	$\Phi(n) \equiv \Phi_i, \quad i = 1, \ldots, n$
Transformation	$\Phi \to e^{iq\theta}\Phi$	$\Phi_i \to \left(e^{iT_a\theta_a}\right)_{ij} \Phi_j$
	with q a real number	with T_a in the irrep of Φ
The gauge field	A_μ	G_μ^a
irrep of the gauge field	$q = 0$	Adjoint
Covariant derivative	$D^\mu = \partial^\mu + igqA^\mu$	$D^\mu = \partial^\mu + igT_aG_a^\mu$
Field strength tensor	$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$	$G_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - gf_{abc}G_b^\mu G_c^\nu$
Gauge kinetic term	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$	$-\frac{1}{4}G_{\mu\nu a}G_a^{\mu\nu}$
Gauge boson self-interactions	No	Yes
Sign of the beta function	Positive	Can assume either sign

4.5 Summary

We summarize the main differences between the Abelian and non-Abelian cases in Table 4.1.

For further reading

QFT aspects of non-Abelian symmetries can be found, for example, in Chapters 15 and 16 of Ref. [2]. More on running coupling constants can be found in, for example, Chapter 23 of Ref. [15].

Problems

Question 4.1: Algebra

1. Insert Eq. (4.17) into Eq. (4.19) and use the $SU(N)$ algebra to obtain Eq. (4.20).
2. Use the definition in Eq. (4.21) to derive Eq. (4.23).
3. Derive the Feynman rules for the self-interactions of the gauge fields.

Question 4.2: Constructing invariants

Consider the following model:

(i) the symmetry is a global

$$SU(3) \times SU(2) \times U(1). \tag{4.27}$$

(ii) There are three complex scalars:

$$A(3,2)_x, \qquad B(T,3)_{+2}, \qquad C(\bar{3},2)_{+1}. \tag{4.28}$$

(iii) There are no fermions.

1. Find all possible values of x and of T such that the term ABC is allowed.
2. Repeat the question for a term of the form $ABC^\dagger$.

Question 4.3: $SU(3)$ and a flavor symmetry

Consider the following model:

(i) The symmetry is a global

$$SU(3)_G. \tag{4.29}$$

(ii) There are N complex scalar fields that transform as

$$\phi_i(3), \qquad i = 1, ..., N. \tag{4.30}$$

(iii) There are no fermions.

We call the index i "a flavor index"

1. What is the number of real scalar DoF in this model?

2. Take $N = 3$. Write the most general renormalizable Lagrangian. For each of the quadratic, cubic, and quartic terms, state explicitly whether it has to have a special structure (symmetric or antisymmetric or hermitian) in flavor space.

3. Take again the $N = 3$ case. Impose an additional global symmetry that we call flavor $SU(3)_F$, such that the three flavors constitute an $SU(3)_F$-triplet $\Phi = (\phi_1, \phi_2, \phi_3)$. In other words, the scalars of this model are in the $(3, 3)$ representation of $SU(3)_G \times SU(3)_F$. What are the implications of imposing the $SU(3)_F$ symmetry on the parameters of $\mathcal{L}$?

Question 4.4: Scalar Lagrangians

Consider a model with four real scalar fields ϕ_i $(i = 1, \ldots, 4)$.

1. Write the kinetic terms for this model. Argue that $\mathcal{L}_{\text{kin}}$ has an $SO(4)$ symmetry where the four real scalars form the fundamental representation of $SO(4)$.

2. Impose an $SO(4)$ symmetry, with the four real scalars in the fundamental representation of $SO(4)$, that is, they form a vector in 4-dimensional real space, $\Phi = (\phi_1, \ldots, \phi_4)^T$. Write the most general renormalizable $\mathcal{L}_{SO(4)}$. Explain why the mass terms and the interaction terms must be proportional to unit matrix in that space.

3. Impose a $U(1)$ symmetry (instead of the $SO(4)$ above, not in addition). To do so, pair the four real scalar fields into two complex scalar fields, $\phi_a = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi_b = (\phi_3 + i\phi_4)/\sqrt{2}$. The $U(1)$ charges are $q_a = +1$ and $q_b = +4$. Write the most general renormalizable $\mathcal{L}_{U(1)}$.

4. $\mathcal{L}_{U(1)}$ has an accidental symmetry. What is it and what are the charges of ϕ_a and ϕ_b under it?

5. Write a non-renormalizable term that breaks the accidental symmetry.

Question 4.5: Discrete sub-symmetries

Consider the following Lagrangian for two Dirac fermions:

$$\mathcal{L} = \overline{\psi}_i[(i\partial\!\!\!/ - m)\delta_{ij}\psi_j, \tag{4.31}$$

where the flavor indices i, j run from 1 to 2. This Lagrangian is manifestly invariant under some discrete symmetries in flavor space. These symmetries, however, are part of the $U(2)$ flavor symmetry under which (ψ_1, ψ_2) form a doublet. For example, the symmetry operation $\psi_i \to -\psi_i$ can be generated by the $U(1)$ transformation: $U = e^{i\alpha}$ with $\alpha = \pi$. Find the $U(2) = SU(2) \times U(1)$ rotations for the following two symmetries:

$$\begin{aligned} &(i) \qquad \psi_1 \to \psi_1, \quad \psi_2 \to -\psi_2 \\ &(ii) \qquad \psi_1 \leftrightarrow \psi_2\,. \end{aligned} \tag{4.32}$$

Hints: You can write your answer as a series of several rotations. The following identity can be useful

$$\exp(i\alpha\sigma_a) = I\cos\alpha + i\sigma_a \sin\alpha \tag{4.33}$$

Question 4.6: Non-Abelian gauge bosons

In this question you are asked to prove that the gauge bosons belong to the adjoint representation. Consider a field ϕ that transforms as an M-dimensional representation, R:

$$\phi \to U\phi, \qquad U = e^{iT_a\theta_a}, \tag{4.34}$$

where a runs from one to the dimension of the group (for $SU(N)$, $a = 1, 2, \ldots, N^2 - 1$) and U and T^a are $M \times M$ matrices. We take θ_a to be independent of x_μ. This may seem weird, as the whole reason to introduce the gauge fields is to let θ_a depend on x_μ. Yet, once the gauge fields are introduced, they must transform also under the global symmetry, so $\theta_a = \text{const}(x_\mu)$ is just a special case. The covariant derivative is

$$D_\mu = \partial_\mu + igG_\mu, \qquad G_\mu \equiv G^a_\mu T_a. \tag{4.35}$$

1. Show that the infinitesimal transformation is

$$U = 1 + iT_a\theta_a + O(\theta_a^2). \tag{4.36}$$

2. Write the infinitesimal transformation of ϕ_k explicitly with the group indices.

3. In order to promote a global symmetry to a local one, $D_\mu\phi$ must transform the same way as ϕ, that is,

$$D_\mu\phi \to UD_\mu\phi. \tag{4.37}$$

Show that Eq. (4.37) implies that

$$G_\mu \to UG_\mu U^\dagger - \frac{1}{g}T_a\partial_\mu\theta_a. \tag{4.38}$$

4. Show that Eq. (4.38), together with the algebra of the group, imply that for an infinitesimal gauge transformation

$$G_\mu^a \to G_\mu^a + \theta^c f^{abc} G_\mu^b - \frac{1}{g}\partial_\mu\theta^a. \tag{4.39}$$

5. Using Eq. (4.36), argue that the irrep of the gauge field is

$$(T^c)^{ab} = -if^{cab}. \tag{4.40}$$

Using this result argue that the gauge fields transform as the adjoint.