Chapter 1

Lagrangians

In this chapter, we review the basic tools that we use in the book. In particular, we introduce the Lagrangian and present some simple Lagrangians involving scalar and fermion fields.

1.1 Introduction

Modern physics encodes the basic laws of nature in the action S, and postulates the principle of minimal action in its quantum interpretation. In Quantum Field Theory (QFT), the action is an integral over spacetime of the “Lagrangian density” or Lagrangian, $\mathcal{L}$, for short. For most of our purposes, it is enough to consider the Lagrangian, rather than the action. In this chapter we explain how Lagrangians are “constructed”. Later in the book we discuss how the numerical values of the parameters that appear in the Lagrangian are determined, and how to test if a Lagrangian provides a viable description of nature.

The QFT equivalent of the generalized coordinates of classical mechanics are the fields. The action is given by

$$S = \int d^4x \, \mathcal{L} \tag{1.1}$$

where $d^4x = dx^0 dx^1 dx^2 dx^3$ is the integration measure in four-dimensional Minkowski space. In general, we require the following properties for the Lagrangian:

(i) It is a function of the fields and their derivatives only.

(ii) It depends on the fields taken at one spacetime point x^μ only, leading to a local field theory.

(iii) It is real, so that the total probability is conserved.

(iv) It is invariant under the Poincaré group, that is under spacetime translations and Lorentz transformations.

(v) It is an analytic function in the fields. This is not a general requirement, but it is common to all field theories that are solved via perturbation theory. In these cases, we expand around a minimum, and this expansion means that we consider a Lagrangian that is a polynomial in the fields.

(vi) It is invariant under certain internal symmetry groups. The invariance of S (or of $\mathcal{L}$) is in correspondence with conserved quantities and reflects basic symmetries of the physical system.

(vii) Every term in the Lagrangian that is not forbidden by a symmetry should appear.

We often impose an additional requirement:

(viii) Renoarmalizability. A renormalizable Lagrangian contains only terms that are of dimension less than or equal to four in the fields and their derivatives.

The requirement of renormalizability ensures that the Lagrangian contains at most two ∂_μ operations, and leads to classical equations of motion that are no higher than second order derivatives. If the full theory of nature is described by a QFT, its Lagrangian should indeed be renormalizable. The theories that we consider, however, and, in particular, the SM, are only low energy effective theories, valid up to some energy scale Λ. Therefore, we must include also non-renormalizable terms. These terms have coefficients with inverse mass dimensions, $1/\Lambda^n$, $n = 1, 2, \ldots$. For most purposes, however, the renormalizable terms constitute the leading terms in an expansion in E/Λ, where E is the energy scale of the physical processes under study. Therefore, the renormalizable part of the Lagrangian is a good starting point for our study. Thus, in Chapters 1-10, we consider only renormalizable Lagrangians, unless otherwise explicitly stated. In Chapters 11-15, where we describe searches for physics beyond the SM, we consider also non-renormalizable Lagrangians.

Properties (i)–(v) are not the subject of this book. You should be familiar with them from your QFT course(s). We do, however, deal intensively with the other requirements. Actually, the most important message that we would like to convey is the following: *(Almost) all experimental data for elementary particles and their interactions can be explained by the standard model of a spontaneously broken $SU(3) \times SU(2) \times U(1)$ gauge symmetry.*[1]

Writing down a specific Lagrangian is the endpoint of the process known as "model building," and the starting point for a phenomenological interpretation and experimental testing. In this book we explain both sides of this modern way of understanding high energy physics.

1.2 Examples of simple Lagrangians

We next present a few examples of simple Lagrangians of scalar and fermion fields. They are simple in the sense that we do not yet impose any internal symmetry. We use $\phi(x)$ for a scalar

[1] Actually, the great hope of the high-energy physics community is to prove this statement wrong, and to find an even more fundamental theory.

field and $\psi(x)$ for a fermion field. When we consider vector fields, as is first done in section 2.2, we use $A(x)$ for a vector field. We do not consider higher spin fields, as it is not simple to construct a QFT where they are fundamental.

Two comments are in order:

- All fields that we consider are functions of spacetime coordinates, $\phi(x)$, $\psi(x)$, $A(x)$. We leave this spacetime dependence implicit, except in cases where it is relevant.
- We use the notations ϕ, ψ and A for the discussion of generic cases. When we refer to specific cases, we use a different notation. For example, for the electron field, we use the notation e instead of the generic ψ.

1.2.1 Scalars

The most general renormalizable Lagrangian for a single real scalar field ϕ is given by

$$\mathcal{L}_S = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 - \frac{\eta}{2\sqrt{2}}\phi^3 - \frac{\lambda}{4}\phi^4 \,. \tag{1.2}$$

We emphasize the following points:

1. The term with derivatives is called the kinetic term. It is necessary if we want ϕ to be a dynamical field, namely to be able to describe propagation in spacetime.
2. The terms without derivatives are collectively denoted by $-V_\phi$. We then write $\mathcal{L}_S = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V_\phi$, and V_ϕ is called the scalar potential.
3. We work in the "canonically normalized" basis where the coefficient of the kinetic term is 1/2. (This is true for a real scalar field. For a complex scalar field, the canonically normalized coefficient of the kinetic term is 1.)
4. From here on, throughout the book, when we say "the most general Lagrangian", we refer to a Lagrangian where the kinetic terms are canonically normalized, but the other terms are written in a general basis. (Question 2.8 shows that there is no loss of generality in working in the canonically normalized basis.)
5. We do not write a constant term since it does not enter the equation of motion for ϕ.
6. We do not write a term linear in ϕ because, when expanding around a minimum, the linear term vanishes.
7. The quadratic term (ϕ^2) is a mass-squared term. (It is often called simply "a mass term.")
8. The trilinear (ϕ^3) and quartic (ϕ^4) terms describe interactions.
9. Terms with five or more scalar fields (ϕ^n, $n \geq 5$) are non-renormalizable.

1.2.2 Fermions

The basic fermion fields are two component Weyl fermions, ψ_L and ψ_R, where L and R denote, respectively, left-handed and right-handed chirality. Each of ψ_L and ψ_R has two degrees of freedom and is a complex field. They are related to the four-component Dirac field ψ via

$$\psi_R = P_R\psi \equiv \frac{1+\gamma_5}{2}\psi, \qquad \psi_L = P_L\psi \equiv \frac{1-\gamma_5}{2}\psi. \tag{1.3}$$

It is useful to define related left-handed Weyl fermion ψ_R^c and a right-handed Weyl fermion ψ_L^c via

$$\psi_R^c = C\,\overline{\psi_R}^T, \qquad \psi_L^c = C\,\overline{\psi_L}^T, \tag{1.4}$$

where C is the charge conjugation matrix. (The reason for this name becomes clear once we define charge in Chapter 2.)

The most general renormalizable Lagrangian for a single left-handed fermion field ψ_L and a single right-handed fermion field ψ_R is given by

$$\mathcal{L}_F = i\overline{\psi_L}\partial\!\!\!/\psi_L + i\overline{\psi_R}\partial\!\!\!/\psi_R - \left(\frac{m_{MR}}{2}\overline{\psi_R^c}\,\psi_R + \frac{m_{ML}}{2}\overline{\psi_L^c}\,\psi_L + m_D\overline{\psi_L}\psi_R + \text{h.c.}\right). \tag{1.5}$$

We emphasize the following points:

1. The derivative terms are the kinetic terms. They are necessary if we want $\psi_{L,R}$ to be dynamical fields.

2. We work in the canonically normalized basis where the coefficient of the kinetic term is 1.

3. Terms with an odd number of fermion fields violate Lorentz symmetry, and so they are forbidden.

4. The quadratic terms are mass terms. The m_M terms are called Majorana masses and the m_D terms are called Dirac masses.

5. The relative factor of $1/2$ between Majorana and Dirac mass terms is the analog of the similar factor between the mass-squared terms for a real and complex scalar fields.

6. Terms with four or more fermion fields are non-renormalizable.

7. Given the fact that Majorana mass terms are made out of a pair of identical fields, we often write

$$\frac{m_{MR}}{2}\overline{\psi_R^c}\,\psi_R \to \frac{m_{MR}}{2}\psi_R\,\psi_R. \tag{1.6}$$

In case that the Majorana masses vanish, $m_{ML} = m_{MR} = 0$, $\mathcal{L}_F$ can be written in terms of the Dirac fermion field ψ:

$$\mathcal{L}_F(m_M = 0) = i\overline{\psi}\partial\!\!\!/\psi - m_D\overline{\psi}\psi. \tag{1.7}$$

Since ψ_L and ψ_R are different fields, there are four degrees of freedom with the same mass, m_D. In contrast, if the Majorana masses do not vanish, there are generally only two degrees of freedom that have the same mass. In Section 2.1.5 we discuss these issues in more detail, and explain why it is often the case that Majorana masses vanish.

1.2.3 Fermions and scalars

Consider the case of a single left-handed fermion ψ_L, a single right-handed fermion ψ_R and a single real scalar field ϕ. The Lagrangian includes, in addition to terms that involve only the scalar, Eq. (1.2), and terms that involve only the fermions, Eq. (1.5), terms that involve both the scalar and the fermions. They can be obtained by replacing the mass parameters for the fermions with a coupling times the scalar field:

$$-\mathcal{L}_{\rm Yuk} = \frac{Y}{\sqrt{2}}\phi\overline{\psi_L}\psi_R + \frac{Y_{MR}}{2}\phi\overline{\psi_R^c}\,\psi_R + \frac{Y_{ML}}{2}\phi\overline{\psi_L^c}\,\psi_L + {\rm h.c.} \tag{1.8}$$

These terms are called Yukawa interactions. The Y parameters are dimensionless and are called the Yukawa couplings. Note that in Eq. (1.8) we quote $-\mathcal{L}$. This is a common practice when we do not write the kinetic terms.

1.3 Symmetries

We always seek deeper reasons for the laws of nature that have been discovered. These reasons are often closely related to symmetries. The term *symmetry* refers to an invariance of the equations that describe a physical system. The fact that a symmetry and an invariance are related concepts is obvious enough — a smooth ball has a spherical symmetry and its appearance is invariant under rotation.

Symmetries are built into physics as invariance properties of the Lagrangian. If we construct our theories to encode various empirical facts and, in particular, the observed conservation laws, then the equations turn out to exhibit certain invariance properties. For example, if we want to implement energy conservation into the theory, then the Lagrangian must be invariant under time translations (and therefore cannot depend explicitly on time). From this point of view, the conservation law is the input and the symmetry is the output.

Conversely, if we take the symmetries to be the fundamental rules, then various observed features of particles and their interactions are a necessary consequence of the symmetry principle. In this sense, symmetries provide an explanation of these features. In modern particle physics (and, in particular, in this book), we often take the latter point of view, in which symmetries are the input and conservation laws are the output.

In the following we discuss the consequences of *imposing* a symmetry on a Lagrangian. This is the starting point of model building in particle physics: One defines the basic symmetries and the field content, and then obtains the predictions that follow from these imposed symmetries.

There are, however, symmetries that are not imposed and are called *accidental* symmetries. They are outputs of the theory rather than external constraints. Accidental symmetries arise due to the fact that we truncate our Lagrangian. In particular, the renormalizable terms in the Lagrangian often have accidental symmetries that are broken by non-renormalizable terms. Since we study mostly renormalizable Lagrangians, we will often encounter accidental symmetries.

There are various types of symmetries. First, we distinguish between *spacetime* and *internal* symmetries. Spacetime symmetries include the Poincaré group of translations, rotations and boosts. They give the energy–momentum and angular momentum conservation laws. As mentioned above, we always impose this symmetry. The list of possible spacetime symmetries includes, in addition, space inversion (also called parity) P, time-reversal T, and charge conjugation C. (While C is not truly a spacetime symmetry, the way it acts on fermions and the CPT theorem make it simpler to include C in the same class of operators.) The discrete spacetime symmetries are usually covered in QFT courses, but for completeness we discuss them briefly in Appendix 1.A.

Internal symmetries act on the fields, not directly on spacetime. In other words, they act in internal spaces which are mathematical spaces that are generated by the fields. These are the kind of symmetries that we discuss in detail. In Chapter 2 we introduce Abelian symmetries. In Chapter 4 we introduce non-Abelian symmetries.

1.4 Model building

As stated above, writing a Lagrangian is the endpoint of model building. Our procedure of constructing Lagrangians goes as follows. We start by defining the following inputs:

(i) The symmetry;

(ii) The transformation properties of the various scalar and fermion fields under the symmetry operation.

Then we write down the most general Lagrangian that depends on the fields and is invariant under the symmetry.

A renormalizable Lagrangian (or a non-renormalizable one truncated at a certain order) has a finite number of parameters. For a theory with N parameters, we need to perform N appropriate measurements such that additional measurements, from the $(N+1)$'th and on, test the theory. In principle, we do not really need to determine the values of the parameters, and just use experimental inputs to make predictions. In practice, however, it is usually convenient to use the N measurements to determine the values of the Lagrangian parameters and use these parameters to make further predictions. It is important to remember that the values of the parameters are not inputs to model building.

At this point, the above procedure may seem abstract, but it becomes clear and concrete as we work out examples. Along this book, we repeat the process of model building several times. We see how QED (the theory of electromagnetic interactions), QCD (the theory of strong interactions), the LSM (the theory of electroweak interactions among leptons), and the SM itself, can be understood in this way of thinking, starting from a postulate of symmetry principles.

For further reading

There are many books that discuss in detail the QFT-related aspects relevant to our book. For example, some of the standard textbooks are by Peskin and Schroeder [2], Zee [13], Srednicki [14], and Schwartz [15]. Other textbooks that explain many of the relevant issues include Ramond [16], Dine [17], Nagashima [18, 19], and Petrov and Blechman [20].

With regard to some specific points, we mention the following sources:

- For a formal discussion of C and P see, for example, Section 3.6 of Ref. [2], or Sections 11.4-11.6 of Ref. [15].
- For a discussion of the issues with quantizing theories with higher-spin fields see, for example, Ref. [21].
- For discussion of Majorana fermions see, for example, Section 11.3 in Ref. [15].
- For the CPT theorem see, for example, Ref. [22].

Appendix

1.A Discrete spacetime symmetries: C, P and T

The discrete spacetime symmetries, C, P, and T, play an important role in our understanding of nature. Each of these three symmetries has been experimentally shown to be violated in nature, as discussed in detail further below. The CPT combination seems, however, to be an exact symmetry of nature. On the experimental side, no sign of CPT violation has been observed. On the theoretical side, CPT must be conserved for any Lorentz invariant local field theory. Since we only consider such theories, we assume that CPT holds. In this case, CP and T are equivalent. Thus, we usually refer to CP.

1.A.1 C and P

We only consider C and P in theories that involve fermions. Under C, particles and antiparticles are interchanged by conjugating all internal quantum numbers, *e.g.*, reversing the sign of the electromagnetic charge, $Q \to -Q$. Under P, the handedness of space is reversed, $\vec{x} \to -\vec{x}$, and the chirality of fermion fields is reversed, $\psi_L \leftrightarrow \psi_R$. For example, a left-handed (LH) electron e_L^- transforms under C into a LH positron e_L^+, and under P into a right-handed (RH) electron e_R^-.

1.A.2 CP violation and complex couplings

The CP transformation combines charge conjugation C with parity P. For example, a LH electron e_L^- transforms under CP into a RH positron, e_R^+. CP is a good symmetry if there is a basis where all the parameters of the Lagrangian are real. We do not prove it here but provide a simple intuitive explanation of this statement.

Consider a theory with a single complex scalar, ϕ, and two sets of N fermions, ψ_L^i and ψ_R^i ($i = 1, 2, ..., N$) (we define a complex scalar in Chapter 2). The Yukawa interactions are given by

$$-\mathcal{L}_{\text{Yuk}} = Y_{ij}\overline{\psi_{Li}}\phi\psi_{Rj} + Y_{ij}^*\overline{\psi_{Rj}}\phi^\dagger\psi_{Li}, \tag{1.9}$$

where we write the two hermitian conjugate terms explicitly. The CP transformation of the fields is defined as follows:

$$\phi \to \phi^\dagger, \qquad \psi_{Li} \to \overline{\psi_{Li}}, \qquad \psi_{Ri} \to \overline{\psi_{Ri}}. \tag{1.10}$$

Therefore, a CP transformation exchanges the operators

$$\overline{\psi_{Li}}\phi\psi_{Rj}\longleftrightarrow\overline{\psi_{Rj}}\phi^\dagger\psi_{Li}, \tag{1.11}$$

but leaves their coefficients, Y_{ij} and Y_{ij}^*, unchanged. This means that CP is a symmetry of $\mathcal{L}$ if $Y_{ij}=Y_{ij}^*$.

In practice, things are more subtle, since one can define the CP transformation in a more general way than Eq. (1.10):

$$\phi\rightarrow e^{i\theta}\phi^\dagger,\qquad \psi_L^i\rightarrow e^{i\theta_{Li}}\overline{\psi_L^i},\qquad \psi_R^i\rightarrow e^{i\theta_{Ri}}\overline{\psi_R^i}, \tag{1.12}$$

with $\theta,\theta_{Li},\theta_{Ri}$ convention-dependent phases. Then, there can be complex couplings, yet CP would be an exact symmetry. The correct statement is that CP is violated if, using all freedom to redefine the phases of the fields, one cannot find any basis where all couplings are real.

Problems

Question 1.1: Algebra

1. Draw the Feynman diagrams for the interaction terms in the Lagrangian of Eq. (1.2).
2. Starting from Eq. (1.5) and using Eq. (1.3), derive Eq. (1.7).
3. Draw the Feynman diagrams for the Yukawa interaction terms in the Lagrangian of Eq. (1.8).

Question 1.2: Using Natural units

In high energy physics, since relativity and quantum mechanics are essential, it is convenient to use units where

$$\hbar \approx 6.58 \times 10^{-22}\,\mathrm{MeV\,s} = 1, \qquad c \approx 3 \times 10^{8}\,\mathrm{m\,s^{-1}} = 1\,, \qquad \hbar c \approx 2 \times 10^{-13}\,\mathrm{MeV\,m} = 1\,. \tag{1.13}$$

One can think of this convention as a choice of a unit system where the basis is $\{\hbar,\, c,\, \mathrm{eV}\}$ instead of, for example, the {cm, g, sec} of the cgs system. In addition, it is common to make the factors of $\hbar$ and c implicit and measure everything in powers of eV. We reinstate the factors of $\hbar$ and c only when converting to a different unit system. The aim of this exercise is that you gain some practice in using these *natural units.*

1. The width of a particle is defined as the inverse of its lifetime. The mean lifetime for the B^+ meson is $\tau \approx 1.64 \times 10^{-12}$ s. What is its width in eV?
2. Consider a particle with a width of $\Gamma = 2.3$ eV. Recall that in the lab frame $t = \gamma\tau$. What is the average distance that such a particle travels with $\gamma = 100$ before decaying (since $\gamma \gg 1$ you can use $\beta \approx 1$.)
3. Quantum gravity effects cannot be neglected at very short distances. This happens when the energy scale is of the order of the Planck mass,

$$M_{\mathrm{Pl}} \equiv \sqrt{\frac{\hbar c}{G_N}}, \tag{1.14}$$

where G_N is the Newtonian gravitational constant. (The Planck scale constitutes an upper bound on the cut-off scale of all QFTs relevant to Nature.) Express $M_{\rm Pl}$ in GeV, and the Planck length, $L_{\rm Pl} \equiv M_{\rm Pl}^{-1}$, in cm.

4. In oscillation experiments for neutrinos, it is important to know the oscillation length, $L_{\rm osc} = 4\pi E/\Delta m^2$, where Δm^2 is the mass-squared difference between the two neutrino states. For an experiment conducted with neutrinos of $E = 1.3$ GeV, find the value of Δm^2 in units of eV^2 that corresponds to $L_{\rm osc} = 140$ meters.

Question 1.3: Dimensions of terms

It is useful to understand what we refer to as the "dimension of operators" or the "dimension of Lagrangian terms." The action has dimensions of angular momentum. Therefore, in the natural unit system, the action is dimensionless, and the Lagrangian has a mass-dimension of four (or, more generally, of the number of spacetime dimensions).

1. Based on the Lagrangians of Eqs. (1.2) and (1.5), show that canonical scalar fields have dimension $d = 1$ and canonical fermion fields have dimension $d = 3/2$.

2. Find the dimensions of the m^2 parameter in Eq. (1.2) and of the m_{MR}, m_{ML}, and m_D parameters in Eq. (1.5).

3. What are the dimensions of η and λ in Eq. (1.2) and of Y in Eq. (1.8)?

Question 1.4: Accidental symmetries

In this question, we study a classical system in order to show examples of accidental symmetries. Consider a classical one-dimensional pendulum of length ℓ. The one degree of freedom can be chosen to be θ, the angle of the pendulum. Then, the Lagrangian is given by

$$L = \frac{m\ell^2\dot{\theta}^2}{2} - mg\ell(1 - \cos\theta), \tag{1.15}$$

Assuming small oscillations ($\theta \ll 1$), we can expand the potential. Keeping only terms up to second-order, we get

$$L = \frac{m\ell^2\dot{\theta}^2}{2} - \frac{mg\ell\theta^2}{2}, \tag{1.16}$$

which is the Lagrangian of a simple harmonic oscillator. It is well known that the frequency of a simple harmonic oscillator does not depend on its amplitude. Below we aim to understand how this result is related to accidental symmetries.

1. Show that the EoM derived from the Lagrangian of Eq. (1.16) is invariant under dilation, $\theta \to \lambda\theta$, for any finite λ. (We are then saying that L of Eq. (1.16) has dilation symmetry, despite the fact that it is only the EoM that is invariant.)

2. Does the Lagrangian of Eq. (1.15) also have dilation symmetry?

3. Expand the Lagrangian of Eq. (1.15) up to $O(\theta^4)$. Show explicitly that the θ^4 term breaks the dilation invariance. Explain why this implies that this symmetry is accidental.

4. Without a formal proof, argue that dilation symmetry implies that the frequency cannot depend on the amplitude.

What we have shown is that the dilation symmetry is accidental and that it is broken by higher order terms.