Part I: First order expansion

1) The final state is obtained by applying the evolution operator on the vacuum:

$$|\Psi\rangle = \hat{U}\,|vac\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}\right)|vac\rangle = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\hat{H}\right)^n |vac\rangle$$

To 1^{st} order in g : $\quad |\Psi^{(1)}\rangle = |vac\rangle + g \sum_k \sqrt{\lambda_k}\; |1\rangle_{s,k} \otimes |1\rangle_{i,k}$

where $\quad |1\rangle_{\alpha,k} = \int d\omega\; f_{\alpha,k}(\omega)\; \hat{a}^\dagger_\alpha(\omega)\; |vac\rangle \qquad (\alpha = s \text{ or } i)$

represents a single-photon Fock state in temporal mode 'k' (signal or idler)

We have : $\quad {}_{\alpha,k}\langle 1|1\rangle_{\alpha',k'} = \int d\omega \int d\omega'\; f^*_{\alpha,k}(\omega)\, f_{\alpha',k'}(\omega')\; \langle vac|\underbrace{\hat{a}_\alpha(\omega)\,\hat{a}^\dagger_{\alpha'}(\omega')}_{\mathbb{1}\cdot\delta_{\alpha\alpha'}\cdot\delta_{\omega\omega'} + \hat{a}^\dagger_{\alpha'}(\omega')\,\hat{a}_\alpha(\omega)}|vac\rangle$

$\alpha, \alpha' \in \{s,i\}$

$$= \delta_{\alpha\alpha'} \cdot \underbrace{\int d\omega\; f^*_{\alpha,k}(\omega)\, f_{\alpha,k'}(\omega)}_{\delta_{kk'}}\; \underbrace{\langle vac|vac\rangle}_{1}$$

$$= \delta_{\alpha\alpha'}\,\delta_{kk'}$$

<u>Remark</u>: $\left|\langle\Psi^{(1)}|\Psi^{(1)}\rangle\right|^2 = 1 + g^2 \sum_k \lambda_k = 1 + g^2 \simeq 1$ to first order in g → normalized.

Note: the subscript *alpha*=*s*,*i* labels both polarisation and spatial mode (e.g. under type-II phase matching condition *s* and *i* are orthogonally polarised modes), while *k* labels the temporal modes appearing in the Schmidt decomposition.

When $\quad \alpha \neq \alpha'$: $\quad {}_{\alpha,\ell}\langle 1|1\rangle_{\alpha',k} \propto \langle vac|\hat{a}_\alpha \hat{a}^\dagger_{\alpha'}|vac\rangle = \langle vac|\hat{a}^\dagger_{\alpha'}\hat{a}_\alpha|vac\rangle$

$$= 0$$

Therefore, we confirmed that these kets form an orthonormal set. In fact, they form a basis for expressing all single-photon-pair states produced by this SPDC source.

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2) The state with one photon pair in mode 'k' is $\quad |1\rangle_{s,k} \otimes |1\rangle_{i,k} \equiv |1,1\rangle_k$

Thus $\quad P^{(1)}_k = \|{}_k\langle 1,1|\Psi^{(1)}\rangle\|^2 = \left|{}_k\langle 1,1|vac\rangle + g\sum_\ell \sqrt{\lambda_\ell}\; {}_{s,k}\langle 1|1\rangle_{s,\ell} \otimes {}_{i,k}\langle 1|1\rangle_{s,\ell}\right|^2$

(we consider that $\Psi^{(1)}$ is normalized)

$$= \left|\; 0 + g \sum_\ell \sqrt{\lambda_\ell}\; \delta_{k,\ell}\, \delta_{k,\ell}\right|^2$$

$$= g^2 \lambda_k$$

The probability of emitting one photon pair in temporal mode k is $\boxed{g^2 \cdot \lambda_k}$

(to lowest order in g)

3) We obtain the overall probability of emitting a photon pair, we sum over k:

$$\boxed{P^{(1)}_{tot} = \sum_k P^{(1)}_k = g^2 \sum_k \lambda_k = g^2}$$

(the λ_k's form a probability distribution)

We see that the parameter g^2 fully determines the total probability of pair emission, and is usually called the gain parameter.

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4) The post-selected state containing only photon pairs is obtained by discarding the vacuum component and normalizing by $\frac{1}{g}$:

$$|\tilde{\Psi}^{(1)}\rangle = \sum_k \sqrt{\lambda_k}\, |1\rangle_{s,k} |1\rangle_{i,k}$$

→ This is a pure state, a quantum coherent superposition of photon pairs in distinct temporal modes

Conclusions:

* In general, $|\tilde{\Psi}^{(1)}\rangle$ is mode-entangled, as it cannot be written as $|\tilde{\Psi}^{(1)}\rangle = |\tilde{\Psi}^{(1)}\rangle_s \otimes |\tilde{\Psi}^{(1)}\rangle_i$

↳ Only when $\lambda_1 = 1$ (Schmidt number $K=1$) is $|\tilde{\Psi}^{(1)}\rangle$ separable.

Even in this case, the full state (to first order in g) is entangled:

$$|\Psi^{(1)}\rangle = |0\rangle_s \otimes |0\rangle_i + g\, |1\rangle_s \otimes |1\rangle_i \neq |\Psi^{(1)}\rangle_s \otimes |\Psi^{(1)}\rangle_i$$

but measuring entanglement in the $|0\rangle, |1\rangle$ photon basis is challenging (a projective measurement of the vacuum requires 100 % collection and detection efficiency!)

Ref:

https://www.nature.com/articles/ncomms6584

Observing optical coherence across Fock layers with weak-field homodyne detectors (2014)

see also for quantum dots: https://www.nature.com/articles/s41566-019-0506-3

5) The marginal "signal" state is obtained by tracing out the "idler" mode, where $\{|1\rangle_{i,k}\}_{k\geq 0}$ forms a basis of the single-photon idler space :

$$\rho_s^{(1)} = \mathrm{Tr}_i\left(|\tilde{\Psi}^{(1)}\rangle\langle\tilde{\Psi}^{(1)}|\right) = \sum_k {}_{i,k}\langle 1|\tilde{\Psi}^{(1)}\rangle\langle\tilde{\Psi}^{(1)}|1\rangle_{i,k}$$

$$= \sum_k \lambda_k \, |1\rangle_{s,k}\langle 1|$$

6) The marginal state is a statistical mixture with probability \lambda_k to find the signal photon in temporal mode k

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7) $$\rho_{ab} = \rho_s^{(1)} \otimes \rho_s^{(1)} = \sum_{k,l} \lambda_k \lambda_l \, |1\rangle_{a,k}\langle 1| \otimes |1\rangle_{b,l}\langle 1|$$

The Hong-Ou-Mandel effect occurs for all terms with *k*=*l*. For terms is *k* \neq *l*, the photons at the two input ports are in different temporal modes, they are distinguishable and don't interfere (therefore no bunching is observed)

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8) Since \rho_{ab} represents a statistical mixture, the overall probability of having indistinguishable photons at both inputs is simply the sum of the corresponding coefficients with *k* = *l*, i.e.:

$$\boxed{P_{HOM} = \sum_k \lambda_k^2 = 1/K}$$ with K the Schmidt number.

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9) Drawbacks: Two copies of the same state are needed for the HOM measurement => Either pumping two exactly identical SPDC sources with the same laser, or delaying half of the photons for a single source and overlapping them with photons emitted by the same source at a later time (caused by a later pump pulse).

Part II: Photon number statistics

1)
$$\hat{U} = \exp\left[\sum_k g\sqrt{\lambda_k}\,\hat{A}_k^\dagger \hat{B}_k^\dagger - H.c.\right]$$

where $\hat{A}_k = \int d\omega\, f_{s,k}(\omega)\,\hat{a}_s(\omega)$ and $\hat{B}_k = \int d\omega\, f_{i,k}(\omega)\,\hat{a}_i(\omega)$

<u>Commutation rules</u> :

$$[\hat{A}_k, \hat{A}_\ell^\dagger] = \iint d\omega\, d\omega'\, f_{s,\ell}^*(\omega') f_{s,k}(\omega)\, \underbrace{[\hat{a}_s(\omega), \hat{a}_s^\dagger(\omega')]}_{\delta_{\omega,\omega'}}$$

$$= \int d\omega\, f_{s,\ell}^*(\omega)\, f_{s,k}(\omega) = \delta_{k,\ell}$$ and same for $[\hat{B}_k, \hat{B}_\ell^\dagger]$

Since the terms within the exponential commute with each other:

$$\boxed{\hat{U} = \bigotimes_k \exp\left[g\sqrt{\lambda_k}\,\hat{A}_k^\dagger \hat{B}_k^\dagger - H.c.\right]}$$ (not true if they did not commute!)

2) We can directly apply the formula with $\theta \to g\sqrt{\lambda_k}$. When acting on $|vac\rangle$ many terms in the exponentials cancel, and we get:

$$|\psi\rangle_k = e^{-\gamma} e^{\Gamma \hat{A}_k^\dagger \hat{B}_k^\dagger} |vac\rangle = \frac{1}{\cosh g\sqrt{\lambda_k}} \sum_{n=0}^{\infty} (\tanh g\sqrt{\lambda_k})^n \frac{\hat{A}_k^{\dagger n}}{\sqrt{n!}} \frac{\hat{B}_k^{\dagger n}}{\sqrt{n!}} |vac\rangle$$

$$= \boxed{\sqrt{1-p_k}\sum_{n=0}^{\infty} \sqrt{p_k}^{\,n}\, |n,n\rangle_k}$$ entangled in photon number

where $p_k \equiv \tanh^2 g\sqrt{\lambda_k}$, and we used $1-\tanh^2 x = \cosh^{-2} x$

and $\hat{a}^{\dagger n}|vac\rangle = \sqrt{n!}\,|n\rangle$

<u>Notation</u> : $|n,n\rangle_k \equiv |n\rangle_{s,k} \otimes |n\rangle_{i,k} \equiv \frac{(\hat{a}_{s,k}^\dagger)^n}{\sqrt{n!}} \otimes \frac{(\hat{a}_{i,k}^\dagger)^n}{\sqrt{n!}} |vac\rangle$

<u>Remark</u> : In the Heisenberg representation, the operators evolve as:

$$\boxed{\begin{aligned} \hat{A}_k &\longrightarrow \cosh(g\sqrt{\lambda_k})\,\hat{A}_k + \sinh(g\sqrt{\lambda_k})\,\hat{B}_k^\dagger \\ \hat{B}_k &\longrightarrow \cosh(g\sqrt{\lambda_k})\,\hat{B}_k + \sinh(g\sqrt{\lambda_k})\,\hat{A}_k^\dagger \end{aligned}}$$

3) The full multimode quantum state out of SPDC is :

$$|\Psi\rangle = \bigotimes_k |\Psi\rangle_k$$

which is a tensor product of states in infinitely many modes, each containing infinitely many terms (Fock states with decreasing amplitudes)

To compare with the results from part I, we expand this tensor product and keep only the terms to lowest order in g, which contain at most one photon pair in total (i.e. over all temporal modes k)

* First, note that when $g \ll 1$ we have $\sqrt{p_k} \approx g\sqrt{\lambda_k}$ and $\sqrt{1-p_k} \approx 1$

* We introduce the notation $|vac\rangle_{\neq k}$ for the vacuum state in all modes different from k, i.e., $|vac\rangle_{\neq k} = \bigotimes_{\ell \neq k} |vac\rangle_i$

↳ A single photon pair in temporal mode k is represented by $|1,1\rangle_k \otimes |vac\rangle_{\neq k}$

Therefore :

$$|\Psi\rangle \underset{g\ll 1}{\simeq} \bigotimes_k \left(\sum_{n=0}^{\infty} (g\sqrt{\lambda_k})^n \, |n,n\rangle_k \right)$$

$$= \bigotimes_k \left(|vac\rangle_k + g\sqrt{\lambda_k}\, |1,1\rangle_k + O(g^2) \right)$$

$$= \left(\bigotimes_k |vac\rangle_k \right) + g \sum_k \sqrt{\lambda_k}\, |1,1\rangle_k \otimes |vac\rangle_{\neq k} + O(g^2)$$

which is indeed the state $|\Psi^{(1)}\rangle$ found in part I.

We note that the original full state was k-mode separable, but mode entanglement emerges when measuring in the single photon pair regime.

4) We trace out the idler mode to find the marginal state of the signal field in temporal mode 'k' :

$$\hat{\varrho}_{s,k} = \mathrm{Tr}_{i,k}\, |\Psi\rangle\langle\Psi|$$

$$= \sum_{n=0}^{\infty} {}_{i,k}\langle n| \left[(1-p_k) \sum_m \sum_\ell \sqrt{p_k}^{\,m+\ell} \, |m\rangle_{s,k}\langle \ell| \otimes |m\rangle_{i,k}\langle \ell| \right] |n\rangle_{i,k}$$

$$= \sum_{n,m,\ell} (1-p_k) \sqrt{p_k}^{\,m+\ell} \, \delta_{n,m}\, \delta_{\ell,n} \, |m\rangle_{s,k}\langle \ell|$$

$$= \boxed{(1-p_k) \sum_{n=0}^{\infty} p_k^{\,n} \, |n\rangle_{s,k}\langle n|}$$

We recognize a thermal state, which is maximally mixed for a harmonic oscillator. The probability to measure n photons is $\boxed{\frac{p_k^{\,n}}{1-p_k}}$, it decreases exponentially with n.

5) We consider the state of the signal field : $\hat{\varrho}_s = (1-p)\sum_{n=0}^{\infty} p^n \, |n\rangle\langle n|$

$$g^{(2)}(0) = \frac{\mathrm{Tr}(\hat{\varrho}\, \hat{A}^\dagger \hat{A}^\dagger \hat{A} \hat{A})}{\mathrm{Tr}(\hat{\varrho}\, \hat{A}^\dagger \hat{A})^2} = \frac{\mathrm{Tr}(\hat{A}\hat{A}\,\hat{\varrho}\,\hat{A}^\dagger \hat{A}^\dagger)}{\mathrm{Tr}(\hat{A}\,\hat{\varrho}\,\hat{A}^\dagger)^2}$$

$$* \quad \mathrm{Tr}(\hat{A}\,\hat{\varrho}\,\hat{A}^\dagger) = \sum_{m=0}^{\infty} \langle m| \left[(1-p) \sum_{n=0}^{+\infty} p^n \, \hat{A}\, |n\rangle\langle n| \hat{A}^\dagger \right] |m\rangle$$

$$= (1-p) \sum_m \sum_n p^n \langle m|\hat{A}|n\rangle\langle n|\hat{A}^\dagger|m\rangle$$

$$= (1-p) \sum_m \sum_n p^n \cdot n \, \langle m|n-1\rangle\langle n-1|m\rangle$$

$$= (1-p) \sum_{n=1}^{\infty} n\, p^n = (1-p)\, p \sum_{n=1}^{\infty} n\, p^{n-1} = \boxed{\frac{p}{1-p}}$$

Consider the power series $f(p) = \sum_{n=0}^{\infty} p^n \rightarrow f'(p) = \sum_{n=1}^{\infty} n\, p^{n-1}$

$$= \frac{1}{1-p} \longrightarrow \quad = \frac{1}{(1-p)^2}$$

* The numerator is found along similar lines:

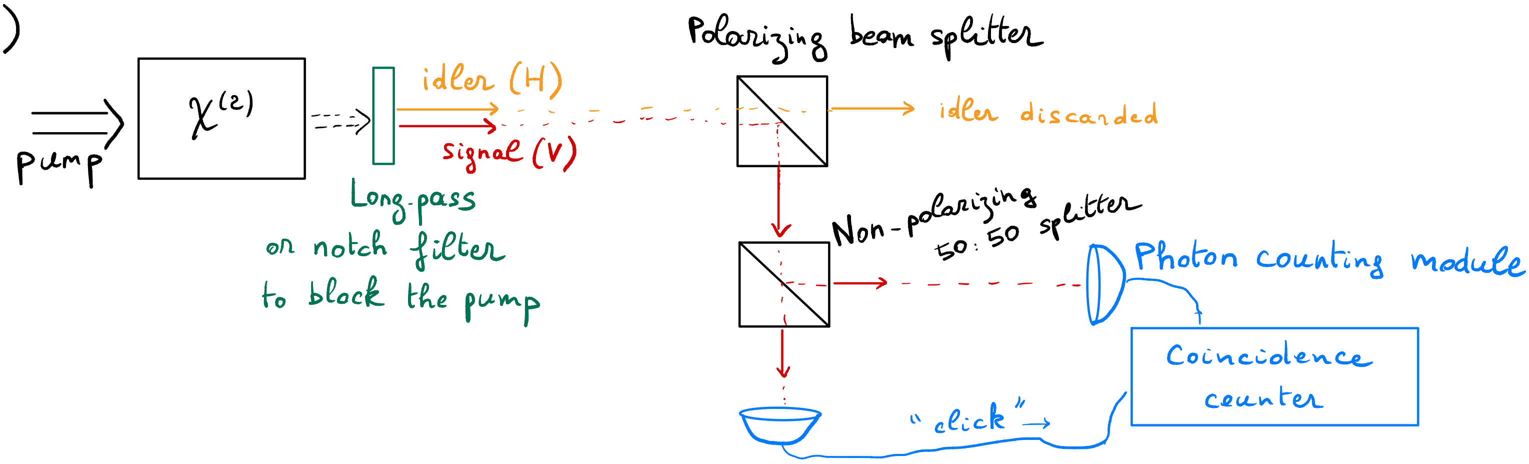
$$\mathrm{Tr}\left(\hat{A}\hat{A}\hat{\rho}\hat{A}^\dagger\hat{A}^\dagger\right) = (1-p)\sum_n n(n-1)p^n = (1-p)p^2\sum_n n(n-1)p^{n-2}$$

We recognize: $f''(p) = \dfrac{2}{(1-p)^3}$ so that:

$$\boxed{\mathrm{Tr}\left(\hat{A}\hat{A}\hat{\rho}\hat{A}^\dagger\hat{A}^\dagger\right) = \frac{2p^2}{(1-p)^2}}$$

and finally: $\boxed{g^{(2)}(0) = \dfrac{2p^2}{(1-p)^2} \times \dfrac{(1-p)^2}{p^2} = 2}$ → super-Poissonian

6)



To a good approximation, g^(2) is given by the ratio of double-coincidence counts to the product of single counts on the two detectors. See this paper for full details: https://doi.org/10.1364/oe.22.003244

7) Going back to the full multimode state, the marginal state of the signal field is:

$$\boxed{\hat{\rho}_S = \bigotimes_k \hat{\rho}_{S,k}} \quad \text{with} \quad \hat{\rho}_{S,k} = (1-p_k)\sum_{n=0}^{\infty} p_k^n \, |n\rangle_k\langle n|$$

(tensor product of many thermal states)

8) When $g \ll 1$, we can make the approximation $p_k = g^2\lambda_k$ $(\tanh x \approx x)$

Then: $\langle \hat{n}_k \rangle = \mathrm{Tr}\left(\hat{A}_k\hat{\rho}_{S,k}\hat{A}_k^\dagger\right) = \dfrac{p_k}{1-p_k} \approx g^2\lambda_k$

and $\boxed{\langle \hat{N} \rangle = \sum_k g^2\lambda_k = g^2}$ since $\sum_k \lambda_k = 1$

9) $$:\hat{N}^2: \;=\; :\Big(\sum_k \hat{A}_k^\dagger \hat{A}_k\Big)^2: \;=\; :\Big(\sum_k \big(\hat{A}_k^\dagger \hat{A}_k\big)^2 + \sum_k \sum_{\ell\neq k} \hat{A}_k^\dagger \hat{A}_k \hat{A}_\ell^\dagger \hat{A}_\ell\Big):$$

$$= \sum_k \big(:\hat{n}_k^2:\big) + \sum_k \sum_{\ell\neq k} \hat{A}_k^\dagger \hat{A}_\ell^\dagger \hat{A}_k \hat{A}_\ell$$

commute

$$= \sum_k \hat{A}_k^\dagger \hat{A}_k^\dagger \hat{A}_k \hat{A}_k + \sum_k \sum_{\ell\neq k} \hat{n}_k \hat{n}_\ell$$

10) From 9: $\langle :\hat{N}^2: \rangle = \sum_k \langle \hat{A}_k^\dagger \hat{A}_k^\dagger \hat{A}_k \hat{A}_k \rangle + \sum_k \sum_{\ell\neq k} \langle \hat{n}_k \hat{n}_\ell \rangle$

The first term was computed in 5) : $\langle :\hat{n}_k^2: \rangle = \dfrac{2p_k^2}{(1-p_k^2)} \approx 2p_k^2 \approx \boxed{2g^4\lambda_k^2}$

For the 2nd term, we use the fact that the full state is mode-separable,

so that: $\boxed{\langle \hat{n}_k \hat{n}_\ell \rangle = \langle \hat{n}_k \rangle \langle \hat{n}_\ell \rangle \text{ if } k\neq \ell}$

Proof : $\langle \hat{n}_1 \hat{n}_2 \rangle = \mathrm{Tr}\big(\hat{\varrho}_1 \hat{n}_1 \otimes \hat{\varrho}_2 \hat{n}_2\big) = \mathrm{Tr}(\hat{\varrho}_1 \hat{n}_1) \times \mathrm{Tr}(\hat{\varrho}_2 \hat{n}_2)$

Therefore: $$\sum_k \sum_{\ell\neq k} \langle \hat{n}_k \hat{n}_\ell \rangle = \sum_k \langle \hat{n}_k \rangle \sum_{\ell\neq k} \langle \hat{n}_\ell \rangle$$

$$= \boxed{g^4 \sum_k \lambda_k (1-\lambda_k)}$$

because $\sum_\ell \lambda_\ell = 1 \Rightarrow \sum_{\ell\neq k} \lambda_\ell = 1 - \lambda_k$

Finally: $$\boxed{\frac{\langle :\hat{N}^2: \rangle}{\langle \hat{N} \rangle^2} = 2\sum_k \lambda_k^2 + \sum_k \lambda_k - \sum_k \lambda_k^2 = 1 + \frac{1}{K}}$$

with $\dfrac{1}{K} = \sum_k \lambda_k^2$