Problem: Photon Statistics in Parametric Down Conversion

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Reminders We have seen that the general form of the Hamiltonian for spontaneous parametric down conversion (when signal and idler fields are in distinct modes, e.g. under type-II phase matching) is

$$\hat{H}_{SPDC} = i\,[\hbar g \int d\omega \int d\omega' f(\omega,\omega')\hat{a_s}^\dagger(\omega)\hat{a_i}^\dagger(\omega') - H.c.] \tag{1}$$

where g parametrizes the effective gain of the process. In this exercise, we will always consider the low gain regime where $g \ll 1$. The joint spectral amplitude f is normalised

$$\int d\omega \int d\omega' |f(\omega,\omega')|^2 = 1 \tag{2}$$

We recall that the Schmidt decomposition consists in writing f as a sum of separable temporal modes for signal and idler fields:

$$f(\omega,\omega') = \sum_{k\geqslant 1} \sqrt{\lambda_k} f_{s,k}(\omega) f_{i,k}(\omega') \tag{3}$$

where

$$\int d\omega f_{\alpha,k}(\omega)^* f_{\alpha,l}(\omega) = \delta_{k,l} \quad \text{with } \alpha = s, i \tag{4}$$

The λ_k's are positive real numbers forming a probability distribution ($\sum_k \lambda_k = 1$) and are by convention labelled in decreasing order ($\lambda_{k+1} \leqslant \lambda_k$).

1 Part I: First order expansion in the gain parameter

1) Write down the biphoton wavefunction $|\psi^{(1)}\rangle$ produced under the action of $\hat{H}_{SPDC}$ (eq. 1) on the vacuum, keeping only the first order term in g in the Taylor expansion. Introduce the kets $|1\rangle_{s,k}$ and $|1\rangle_{i,k}$; give their expressions and show that they form an orthonormal set (corresponding to photons in different temporal modes).

2) Under an idealised projective measurement, what is the probability $P_k^{(1)}$ of measuring one photon pair in mode k?

3) What is the overall probability of emitting a photon pair, $P_{\text{tot}}^{(1)}$?

4) We now assume that our measurement is not sensitive to $|\text{vac}\rangle$, and we write the normalised post-selected state $|\tilde{\psi}^{(1)}\rangle$ for which $P_{\text{tot}}^{(1)} = 1$. Under what condition is $|\tilde{\psi}^{(1)}\rangle$ mode-entangled? And mode-separable?

   When $|\tilde{\psi}^{(1)}\rangle$ is mode-separable, is it also the case for $|\psi^{(1)}\rangle$?

5) Consider the general form of $|\tilde{\psi}^{(1)}\rangle$, and a particular experiment where all information about the idler photon's temporal mode k is either lost or ignored. What is then the marginal state of the signal field, $\rho_s^{(1)}$?

6) Under what conditions is $\rho_s^{(1)}$ pure? Or mixed?

7) We now consider two copies of $\rho_s^{(1)}$, which we call ρ_a and ρ_b, impinging on a beam splitter. Write explicitly the input state ρ_{ab}.

   Which terms inside ρ_{ab} give rise to Hong-Ou-Mandel anti-correlations at the output (i.e., they correspond to indistinguishable photons)? What type of output should we expect from the other terms?

8) Deduce the expression of the probability to observe the Hong-Ou-Mandel effect (bunching of two input photons into one output) as a function of the coefficients λ_k in the Schmidt decomposition, and relate it to the Schmidt number K. Note that this probability represents the visibility of the HOM dip.

9) To conclude this part, comment on the drawbacks of using this technique to measure the Schmidt number of a parametric down conversion source.

2 Part II: Photon number statistics of two mode squeezed state

1) Using the Schmidt decomposition, show that the evolution operator $\hat{U} = \exp\left(-\frac{i}{\hbar}\hat{H}_{SPDC}\right)$ can be factorized in a product of independent two-mode squeezing transformations. You will introduce the operators $\hat{A}_k$ and $\hat{B}_k$ for distinct pairs of temporal modes of signal and idler fields, give their expressions and compute their commutation rules.

2) We now consider one particular pair of temporal modes k and look for the joint state $|\psi\rangle_{k,k}$ at the SPDC output to all orders in the small parameter g. Using the following identity (a correlate of the Baker-Campbell-Hausdorff formula):

$$e^{\theta(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})} = e^{\Gamma\hat{a}^\dagger\hat{b}^\dagger}\, e^{-\gamma(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+1)}\, e^{-\Gamma\hat{a}\hat{b}} \tag{5}$$

   with $\Gamma = \tanh\theta$ and $\gamma = \ln(\cosh\theta)$, find the final state for the pair of modes A_k, B_k (both initially in vacuum state).
   Express the result as a function of the parameter $p_k = \tanh^2 g\sqrt{\lambda_k}$.

3) Write down the full multimode state produced by this SPDC source. Is it mode-entangled? Reconcile this result with that of part I by retaining only the terms to lowest order in g and making the appropriate approximations in the limit $g \ll 1$.

4) We temporarily assume that only one λ_k is non-zero – or equivalently that we are able to perform a temporal mode-selective ('k'-selective) measurement. What is the marginal state of the signal field in mode A_k when information about the idler field is lost or ignored? To what general category of states does it belong? What is the probability of having n signal photons in this mode?

5) Compute the second order correlation (at zero time delay) for the marginal state of the signal field:

$$g_s^{(2)}(0) = \frac{\langle\hat{A}^\dagger\hat{A}^\dagger\hat{A}\hat{A}\rangle}{\langle\hat{A}^\dagger\hat{A}\rangle^2} \tag{6}$$

   where we temporarily dropped the label k for brevity, since we consider a single pair of modes. You may need the power series expansion:

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p} = f(p) \quad \text{for } p < 1 \tag{7}$$

   and its corresponding derivatives $f'(p)$, $f''(p)$.

6) What experimental setup would you build to estimate $g_s^{(2)}(0)$ from a type-II SPDC source? Make a sketch.

7) We now consider again the full multimode state $|\psi\rangle$ resulting from the evolution $\hat{U}$. Write down the marginal state of the signal field $\hat{\rho}_s$ when information about the idler field is lost or ignored.

8) The operator measuring the total photon number in the signal beam is $\hat{N} = \sum_k \hat{A}_k^\dagger \hat{A}_k = \sum_k \hat{n}_k$. In the low gain regime, show that $\langle \hat{N} \rangle \simeq g^2$.

9) The second order correlation function is generalised to the case of detectors that do *not* distinguish between temporal modes (corresponding to most experimental scenarios):

$$g_s^{(2)}(0) = \frac{\langle : \hat{N}^2 : \rangle}{\langle \hat{N} \rangle^2} \tag{8}$$

where : : designates normal ordering of all encompassed operators; e.g. $: \hat{A}\hat{A}^\dagger\hat{A} : = \hat{A}^\dagger\hat{A}\hat{A}$ (creation operators on the left, annihilation operators on the right).

First show that

$$: \hat{N}^2 : = \sum_k \hat{A}_k^\dagger \hat{A}_k^\dagger \hat{A}_k \hat{A}_k + \sum_k \sum_{l \neq k} \hat{n}_k \hat{n}_l \tag{9}$$

10) From there, show that for $g \ll 1$ the second order correlation of the signal field is $g^{(2)}(0) = 1 + 1/K$ with K the Schmidt number (justify all steps in the derivation). Therefore, a practical way to estimate the Schmidt number of a SPDC source is to measure the normalised second order correlation function of the signal (or idler) field alone.

More generally, one can show that a statistical mixture of an increasing number of thermal modes asymptotically displays Poissonian photon statistics, and in particular $g^{(2)}(0) \to 1$ as $K \to \infty$. This result explains why observing bunching in thermal light is very difficult in practice, as it requires isolating a single spatio-temporal mode. Using pulsed SPDC is a rather convenient way to produce thermal light and measure photon statistics with $g^{(2)}(0) \simeq 2$, yet it requires a separable joint spectral amplitude.