Problem Set 10 - The Squeezed vacuum

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1 - $\hat{E}(\phi) = \hat{E}_Q \cos(\phi) + \hat{E}_P \sin(\phi)$

$$\hat{E}_Q = \xi_0 (\hat{a} + \hat{a}^+) \qquad \hat{E}_P = i\xi_0 (\hat{a}^+ - \hat{a})$$

2 - Using $[\hat{a}, \hat{a}^+] = 1$

we find $[\hat{E}_Q, \hat{E}_P] = 2i\xi_0^2$

$$\Rightarrow \Delta E_Q \Delta E_P \geq \xi_0^2$$

3 - • $[\hat{P}, \hat{Q}] = \frac{i}{2}$

$$\Rightarrow \Delta Q \Delta P \geq \frac{1}{4} \qquad (*)$$

• The free-field Hamiltonian is : $\hat{H} = \frac{1}{2} \int_V \frac{D^2}{\varepsilon_0 \varepsilon} + \frac{B^2}{\mu_0} d^3\vec{r}$

We have seen in the lecture that it rewrites as

$$\hat{H} = \frac{\hbar\omega}{2} (\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$$

On the other hand, $\hat{Q}^2 + \hat{P}^2 = \frac{1}{4} \left[(\hat{a}^+ + \hat{a})^2 + i^2 (\hat{a}^+ - \hat{a})^2 \right]$

$$= \frac{1}{2} (\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$$

$$\Rightarrow \hat{H} = \hbar\omega (\hat{Q}^2 + \hat{P}^2)$$

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4 - • $\langle \hat{E}(\phi) \rangle = \langle 0 | \hat{E}(\phi) | 0 \rangle = 0$

$$\langle \hat{E}(\phi)^2 \rangle = \xi_0^2 \langle 0 | (\hat{a} e^{-i\phi} + \hat{a}^+ e^{i\phi})^2 | 0 \rangle = \xi_0^2$$

$$\Rightarrow (\Delta E(\phi))^2 = \xi_0^2$$

• $\Delta n^2 = \langle 0 | \hat{a}^+\hat{a}\,\hat{a}^+ a | 0 \rangle - \langle 0 | \hat{a}^+ \hat{a} | 0 \rangle = 0$

• $\Delta Q^2 = \Delta P^2 = \frac{1}{4} \Rightarrow \Delta Q \Delta P = \frac{1}{4}$

Relation (*) is saturated : $|0\rangle$ is a state of minimum uncertainty.

5_ For state $|m\rangle$,

- $\langle \hat{E}(\phi) \rangle = 0$
  $\langle \hat{E}(\phi)^2 \rangle = \xi_0^2 (2m+1)$ $\Rightarrow$ $\left(\Delta E(\phi)\right)^2 = \xi_0^2 (2m+1)$

- $\langle \hat{m} \rangle = m$
  $\langle \hat{m}^2 \rangle = m^2$ $\Rightarrow$ $\Delta m = 0$

- $\Delta Q^2 = \Delta P^2 = \Delta Q \Delta P = \frac{1}{4}(2m+1)$

State $|m\rangle$ has higher quadrature fluctuations than $|0\rangle$ (but the photon number fluctuations are still 0)

3_ 6_ ~~Erratum~~ : $\hat{H}_I = i\frac{\hbar g}{2}\left((\hat{a}^+)^2 - (\hat{a})^2 \right)$

In order to implement this Hamiltonian with SPDC, we need a $\chi^{(2)}$ nonlinear medium and we need to reach phase matching.

7- We use $[\hat{a}^2, \hat{a}^+] = 2\hat{a}$ and $[\hat{a}^{+2}, \hat{a}] = -2\hat{a}^+$

$$\frac{d\hat{Q}}{dt} = \frac{i}{\hbar}[\hat{H}_I, \hat{Q}] = g(\hat{a}^+ + \hat{a}) = g\hat{Q}$$

$$\frac{d\hat{P}}{dt} = \frac{i}{\hbar}[\hat{H}_I, \hat{P}] = -ig(\hat{a}^+ - \hat{a}) = -g\hat{P}$$

8- $\hat{Q}(t) = e^{gt}\hat{Q}_0$
$\hat{P}(t) = e^{-gt}\hat{P}_0$

9- $\Delta Q(t)^2 = \langle e^{2gt}\hat{Q}_0^2 \rangle - \langle e^{gt}\hat{Q}_0 \rangle^2 = e^{2gt}\Delta Q_0^2$
$\Delta P(t)^2 = e^{-2gt}\Delta P_0^2$
$\Delta Q(t)\Delta P(t) = \Delta Q_0 \Delta P_0 = \frac{1}{4}$ for initial state $|0\rangle$

$\Rightarrow$ Uncertainty in one quadrature is reduced, at the expense of uncertainty increase in the other quadrature. The total uncertainty remains constant.

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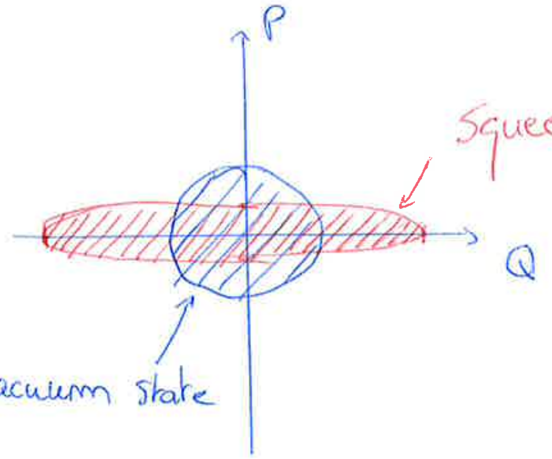

11 - $$\hat{a}^+(t) = \hat{Q}(t) - i\hat{P}(t)$$
$$= e^{gt}\left(\frac{\hat{a}_0 + \hat{a}_0^+}{2}\right) - i\, e^{-gt}\left(i\,\frac{\hat{a}_0^+ - \hat{a}_0}{2}\right)$$
$$\hat{a}^+(t) = \hat{a}_0 \,\mathrm{sh}\, gt + \hat{a}_0^+ \,\mathrm{ch}\, gt$$

Similarly, $\hat{a}(t) = \hat{Q}(t) + i\hat{P}(t)$
$$\hat{a}(t) = \hat{a}_0 \,\mathrm{ch}\, gt + \hat{a}_0^+ \,\mathrm{sh}\, gt$$

$$[\hat{a}(t), \hat{a}^+(t)] = [\hat{a}_0 \,\mathrm{ch} gt, \hat{a}_0^+ \,\mathrm{ch} gt] + [\hat{a}_0^+ \,\mathrm{sh} gt, \hat{a}_0 \,\mathrm{sh} gt]$$
$$= \mathrm{ch}^2 gt - \mathrm{sh}^2 gt$$
$$= 1$$
$\Rightarrow$ Commutation rules are preserved at all times.

12 - $$\langle \hat{n}(t)\rangle = \langle 0|\,\hat{a}^+(t)\,\hat{a}(t)\,|0\rangle$$
$$= \mathrm{sh}^2 gt$$
$$\langle \hat{n}^2(t)\rangle = \langle 0|\,\hat{a}^+(t)\,\hat{a}(t)\,\hat{a}^+(t)\,\hat{a}(t)\,|0\rangle$$
$$= \langle 0|\,a^+(t)a(t)\left(\mathrm{sh}^2 gt\,|0\rangle + \sqrt{2}\,\mathrm{ch}\, gt\,\mathrm{sh}\, gt\,|2\rangle\right)$$
$$= \mathrm{sh}^4 gt + 2\,\mathrm{sh}^2 gt\,\mathrm{ch}^2 gt$$
$$\Rightarrow \Delta n^2 = 2\,\mathrm{sh}^2 gt\,\mathrm{ch}^2 gt$$

Remark: we can write $\Delta n^2 = 2\langle n(t)\rangle\left(1 + \langle n(t)\rangle\right)$
The average photon number $\langle n(t)\rangle$ is no longer 0 for squeezed vacuum! Similarly, photon number fluctuations are now also non-zero.