Problem set 10

Nonlinear Optics for Quantum Technologies

May 1, 2025

The squeezed vacuum

In this problem, we consider the properties of a single mode $(\vec{k}, \vec{E}_k)$ whose electric field amplitude is expressed as

$$\hat{E}(\phi) = \xi_0 \left(\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}\right) \tag{1}$$

with $\xi_0 = \sqrt{\frac{\hbar\omega}{2\epsilon V}}$ and $\phi = \omega t - \vec{k} \cdot \vec{r}$

1 Field quadratures

1. Show that one can rewrite $\hat{E}(\phi) = \hat{E}_Q \cos(\phi) + \hat{E}_P \sin(\phi)$ and give the expressions of the quadrature operators $\hat{E}_Q, \hat{E}_P$.

2. For two operators $\hat{A}$ and $\hat{B}$, the Robertson uncertainty relation imposes

$$(\Delta A)(\Delta B) \geq \frac{1}{2} \left|\langle[\hat{A}, \hat{B}]\rangle\right| \tag{2}$$

   where $(\Delta A)^2 = \langle \hat{A}^2\rangle - \langle \hat{A}\rangle^2$ is the variance characterizing the fluctuations of A.

   Give a lower bound for $\Delta E_Q \times \Delta E_P$, irrespective of the quantum state of light.

3. We now define $\hat{Q} = \frac{\hat{a}^\dagger + \hat{a}}{2}$ and $\hat{P} = i\frac{\hat{a}^\dagger - \hat{a}}{2}$. Compute a lower bound for $\Delta Q \times \Delta P$ and express the free-field Hamiltonian as a function of $\hat{Q}$ and $\hat{P}$. (*Hint:* compute $\hat{Q}^2 + \hat{P}^2$)

2 Vacuum state and number states

4. We consider that the field is in the ground state $|0\rangle$, corresponding to perfect vacuum.

   - Compute the variance of the electric field $(\Delta E(\phi))^2$ in this case, i.e., the vacuum fluctuations.
   - What are the fluctuations in photon number $\hat{n}$?
   - Compute the product $\Delta \times Q\Delta P$ and compare with the lower bound obtained in question 3.

5. Answer the same three questions for the Fock state $|n\rangle$ with n photons. Comment the results.

3 Squeezed vacuum state

We consider a perturbation that can be modeled by the Hamiltonian:

$$\hat{H}_I = i\frac{\hbar g}{2}\left((\hat{a}^\dagger)^2 + (\hat{a})^2\right) \tag{3}$$

where $g \in \mathbb{R}$ for simplicity.

6. In order to implement this Hamiltonian with SPDC, what experimental aspects should be satisfied ?

7. We now work in the Heisenberg picture where the operators evolve in time, $\hat{a}(t)$, and the state remains $|0\rangle$. We know that the free-field Hamiltonian yields a unitary evolution as $e^{-i\omega t}$, which we factor out, and only study the dynamics caused by the perturbation $\hat{H}_I$ introduced above.

   Write the equations of motion for $\hat{Q}(t)$ and $\hat{P}(t)$ under $\hat{H}_I$.

8. Express the solution at time t as a function of $\hat{Q}_0 = \hat{Q}(0)$ and $\hat{P}_0 = \hat{P}(0)$.

9. Compute $\Delta Q(t), \Delta P(t)$ and $\Delta Q(t) \times \Delta P(t)$.

10. Draw the vacuum and squeezed vacuum in phase space.

11. From the results obtained in question 8, express $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ as a function of $\hat{a}_0, \hat{a}_0^\dagger$.

    Show that the commutation rules are preserved at all times.

12. Compute the mean photon number $\langle \hat{n}(t) \rangle$ and standard deviation $\Delta n(t)$ for the squeezed vacuum. Compare the results with those obtained for the standard vacuum.