I) The CHSH inequality

1) In writing down the entire set of measurement results, we assumed that :

- Even unperformed measurements have a result → "realism" hypothesis.

- The choice of which measurement (1 or 2) is performed on either side does not affect the statistics on the other side → "locality"

2) If $A_1^j = A_2^j$ then $|A_1^j + A_2^j| = 2$ and $|A_1^j - A_2^j| = 0$

If $A_1^j \neq A_2^j$ then $|A_1^j + A_2^j| = 0$ and $|A_1^j - A_2^j| = 2$

(same for Bob of course)

3) From question 2), we see that the product can take only three values 0 or ± 4 :

$$(A_1^j \pm A_2^j) \times (B_1^j \pm B_2^j) = 0 \text{ or } \pm 4$$

multiplying by $S(\pm 1; \pm 1) = \pm 1$ preserves this set of possible values.

Therefore $\tilde{S}^j \in \{-4; 0; 4\} \ \forall j$

4) It is immediate that after averaging $-4 \leqslant \langle \tilde{S} \rangle \leqslant 4$

where $\langle \tilde{S} \rangle = \sum_{\alpha,\beta=\pm 1} S(\alpha,\beta)\big(\langle A_1 B_1\rangle + \alpha\langle A_2 B_1\rangle + \beta\langle A_1 B_2\rangle + \alpha\beta\langle A_2 B_2\rangle\big)$

5) With the specific choice $S(-1,-1) = -1$ and $S(1,1) = S(-1,1) = S(1,-1) = 1$

we find:
$$\begin{aligned}\langle \tilde{S} \rangle = &+\big(\langle A_1 B_1\rangle + \langle A_2 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_2\rangle\big)\\
&+\big(\langle A_1 B_1\rangle + \langle A_2 B_1\rangle - \langle A_1 B_2\rangle - \langle A_2 B_2\rangle\big)\\
&+\big(\langle A_1 B_1\rangle - \langle A_2 B_1\rangle + \langle A_1 B_2\rangle - \langle A_2 B_2\rangle\big)\\
&-\big(\langle A_1 B_1\rangle - \langle A_2 B_1\rangle - \langle A_1 B_2\rangle + \langle A_2 B_2\rangle\big)\\
= &\ 2 \times \big(\langle A_1 B_1\rangle + \langle A_2 B_1\rangle + \langle A_1 B_2\rangle - \langle A_2 B_2\rangle\big)\end{aligned}$$

From which we conclude that

$$S_2 \equiv \left| \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \right| \leq 2$$

(CHSH inequality)

Remark : The indices 1, 2 can be permuted at will to give 4 equivalent inequalities (they are just labels for measurement settings).

6) The most obvious strategy is to allow the two particles to communicate with each other, so that they always yield identical results, except for A_2 & B_2 where it is always opposite. Then we would get : $S_2 = 4$ (as we see in part II, quantum mechanics allows $S_2 > 2$ but with the bound $S_2 \leq 2\sqrt{2}$. Interestingly QM is non local, but not so as to allow superluminal communication)

Other possible "loopholes" that might be used to violate the CHSH inequality include the selective non-recording of specific results ; therefore we must invoke a "fair sampling" hypothesis when the detection efficiency is not above some threshold (~70%)

But since the experiments by the team of Ronald Hanson (TU Delft) with NV centers, all loopholes have been closed (locality and detection).

Yet, we cannot exclude that the source and measurement units had become corellated in their common past cone ...

7) If we assume that individual detection outcomes (on each side) are distributed according to a binomial distribution, the spread of S_2 will typically scale as $\frac{1}{\sqrt{N}}$

So after 10'000 coincidence events we expect the statistical error to drop to $\sim 1\%$.

II) Experimental CHSH violation with polarisation entangled photons produced by type II SPDC

1-2) For type II phase matching, signal and idler photons are cross-polarised. Yet they are typically emitted along different directions → this distinguishing information must be erased to obtain an entangled state in polarisation. We must also ensure that they are indistinguishable in their spectral modes by operating at degeneracy. We have seen in lecture 08 a possible implementation.

Note other setups are possible, e.g. cascading two type I processes in two crystals rotated by 90°.

3) The normalised state with one photon pair is $|\tilde{\Psi}\rangle \equiv \frac{1}{\sqrt{2}}\left(\hat{a}_H^\dagger \hat{b}_V^\dagger + \hat{a}_V^\dagger \hat{b}_H^\dagger\right)|\text{vac}\rangle$ so that the searched probability is:

$$|\langle\tilde{\Psi}|\Psi\rangle|^2 = \frac{1}{(1+|\lambda|^2)} \times \frac{|\lambda|^2}{4} \times \left|\langle \text{vac}|\left(\hat{a}_H \hat{b}_V + \hat{a}_V \hat{b}_H\right)\left(\hat{a}_H^\dagger \hat{b}_V^\dagger + \hat{a}_V^\dagger \hat{b}_H^\dagger\right)|\text{vac}\rangle\right|^2$$

$$= \frac{|\lambda|^2}{(1+|\lambda|^2)} \times \frac{1}{4} \times \Big| \langle 0|\hat{a}_H \hat{a}_H^\dagger|0\rangle_{a,H} \times \langle 0|\hat{b}_V \hat{b}_V^\dagger|0\rangle_{b,V} + \langle 0|\hat{a}_V \hat{a}_V^\dagger|0\rangle_{a,V} \times \langle 0|\hat{b}_H \hat{b}_H^\dagger|0\rangle_{b,H} \Big|^2$$

$$= \frac{|\lambda|^2}{(1+|\lambda|^2)} \simeq |\lambda|^2 \text{ when } |\lambda|^2 \ll 1 \quad (\text{low gain})$$

4) The post selected wavefunction is $|\tilde{\Psi}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_a |V\rangle_b + |V\rangle_a |H\rangle_b\right)$

Remark : Beware of this compact notation to avoid errors in calculation and interpretation.

First, this notation is only valid in the subspace with a single photon in each spatial mode a and b.

↳ It impossible to express a state with 2 photons in mode a for example, like $\hat{a}^{+}_{H}\hat{a}^{+}_{V}|vac\rangle$ or $\frac{1}{\sqrt{2}}\hat{a}^{+}_{H}\hat{a}^{+}_{H}|vac$

A more general notation is of the form $|n\rangle_{a,H}$ where n is the Fock state number and the index labels each single spatial + polarisation mode.

With this notation : $$|\tilde{\Psi}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_{a,H}|0\rangle_{a,V}|0\rangle_{b,H}|1\rangle_{b,V} + |0\rangle_{a,H}|1\rangle_{a,V}|1\rangle_{b,H}|0\rangle_{b,V}\right)$$

with tensor products $\otimes$ being implicit : $|n\rangle_i|m\rangle_j = |n\rangle_i \otimes |m\rangle_j$

Then any operator should be written $\hat{O} = \hat{O}_{a,H} \otimes \hat{O}_{a,V} \otimes \hat{O}_{b,H} \otimes \hat{O}_{b,V}$

and $$\langle\hat{O}\rangle_{k,l,m,n} = \langle k|\hat{O}_{a,H}|k\rangle_{a,H} \times \langle l|\hat{O}_{a,V}|l\rangle_{a,V} \times \langle m|\hat{O}_{b,H}|m\rangle_{b,H} \times \langle n|\hat{O}_{b,V}|n\rangle_{b,V}$$

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5) Using the classical meaning of polarisation we have :
$$|V_\alpha\rangle_{a/b} = \cos\alpha\,|V\rangle_{a/b} - \sin\alpha\,|H\rangle_{a,b}$$

6) $$P_{\alpha,\beta} = \left|\left(\cos\alpha\langle V|_a - \sin\alpha\langle H|_a\right)\otimes\left(\cos\beta\langle V|_b - \sin\beta\langle H|_b\right)|\tilde{\Psi}\rangle\right|^2$$
$$= \frac{1}{2}\left|-\cos\alpha\sin\beta - \sin\alpha\cos\beta\right|^2 = \frac{1}{2}\sin^2(\alpha+\beta)$$

7) This probality only depends on the sum of detection angles. (other entangled state would lead to dependence on the difference)

It is a key feature of an entangled state : all information about it concerns <u>joint properties</u>. It is only if we let the system of interest becoming entangled (correlated) with an external system (the environment) that we can obtain information about individual subsystems. In this case we break the original entanglement.

<u>Example</u>: Assume that the polarisation becomes correlated with another degree of freedom (of the photons themselves, or of the environment):

$$|\tilde{\Psi}\rangle \longrightarrow |\Psi'\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_a |V\rangle_b |A\rangle_E + |V\rangle_a |H\rangle_b |B\rangle_E\right)$$

So far it is still an entangled states, including the environment.

But the reduced density matrix of the photon pair is now:

$$\rho'_{ab} = \mathrm{Tr}_E\, \rho' = \langle A|_E\, \rho'\, |A\rangle_E + \langle B|_E\, \rho'\, |B\rangle_E$$

$$= |H\rangle_a |V\rangle_b \langle H|_a \langle V|_b + |V\rangle_a |H\rangle_b \langle V|_a \langle H|_b$$

which is a statistical mixture of $|H\rangle|V\rangle$ and $|V\rangle|H\rangle$ states!

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8) We computed in 6): $P_{\alpha\beta} = \frac{1}{2}\sin^2(\alpha+\beta)$

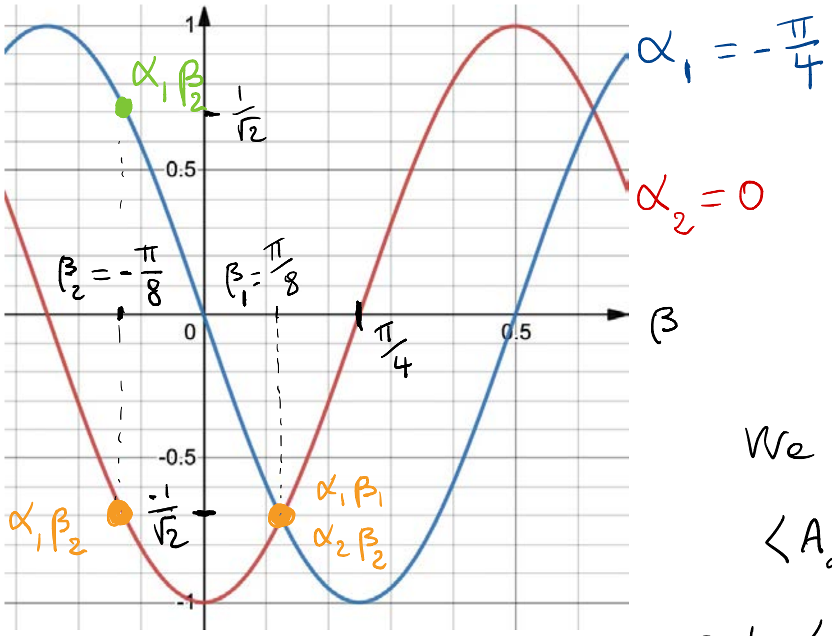
We immediately find that: $P_{\alpha\perp,\beta} = P_{\alpha,\beta\perp} = \frac{1}{2}\cos^2(\alpha+\beta$

and $P_{\alpha\perp,\beta\perp} = P_{\alpha,\beta}$

Therefore: $\langle A_\alpha B_\beta\rangle = \sin^2(\alpha+\beta) - \cos^2(\alpha+\beta) = -\cos(2(\alpha+\beta))$

This form respects the joint dependence in $\alpha+\beta$ only.

Now, the search for angles violating the CHSH inequality is best performed graphically, by plotting $\langle A_\alpha B_\beta\rangle$ vs. β for fixed α



We find that with

$\{\alpha_1 ; \alpha_2\} = \{-\frac{\pi}{4} ; 0\}$

$\{\beta_1 ; \beta_2\} = \{\frac{\pi}{8} ; -\frac{\pi}{8}\}$

We get :

$$\langle A_{\alpha_1} B_{\beta_1} \rangle = \langle A_{\alpha_2} B_{\beta_1} \rangle = \langle A_{\alpha_2} B_{\beta_2} \rangle = -\frac{1}{\sqrt{2}}$$

and $\langle A_{\alpha_1} B_{\beta_2} \rangle = +\frac{1}{\sqrt{2}}$

So that $S_2 = |\langle A_{\alpha_1} B_{\beta_1} \rangle + \langle A_{\alpha_2} B_{\beta_2} \rangle + \langle A_{\alpha_2} B_{\beta_1} \rangle - \langle A_{\alpha_1} B_{\beta_2} \rangle| = 2\sqrt{2} > 2$

9) We conclude from this result, which is experimentally confirmed, that photons produced by SPDC should indeed be described by a pure state consisting of a joint biphoton wavefunction.

When classically we could have thought that photons were randomly emitted into one possible direction of the phase matching cone, they are instead emitted in a single coherent superposition of all possible directions.