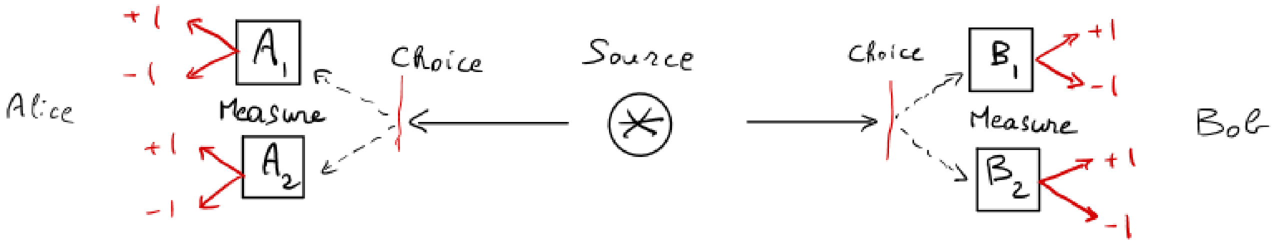
Week 09 - Bell Test

Nonlinear Optics for Quantum Technologies

March 17th, 2025

1 The CHSH inequality

Until John Bell's theorem and its reformulation into experimentally testable inequalities by Clauser, Horne, Shimony and Holt (CHSH) in 1969, the meaning of entangled states was a matter of philosophical debate (EPR paradox and Einstein - Bohr debate). The purpose of this exercies is tho formulate the CHSH inequaltiy (a form of Bell inequality) based on reasonable assumptions of "reality". We consider a very general scenario where a source emits pairs of signals (electrons, photons, etc.) toward two observers (Alice and Bob) who may freely select to perform 1 out of 2 measurements A_1/A_2 for Alice and B_1/B_2 for Bob. The difference between those can be a rotation of measurement basis.



Each measurement result is either +1 or -1 (in case the output has more possible values, or is continuous on can always sort it into two categories labeled ± 1. Now, we assume that if the experiment is repeated N times (each repetition is indexed by j), then any possible outcome can be written as a table of N quartets $\left\{A_1^j, A_2^j, B_1^j, B_2^j\right\}$ each belonging to $\{\pm 1, \pm 1, \pm 1, \pm 1\}$

1. Explicit the physical assumptions that are implicitly needed to justify the last statement above.
2. For each repetition what, are the possible values that the pair $|A_1^j \pm A_2^j|$ can take? (The same applies to $|B_1^j \pm B_2^j|$)
3. What are the possible values for each j of the quantity

$$\tilde{S} = \sum_{\alpha,\beta=\pm 1} S(\alpha,\beta)(A_1^j + \alpha A_2^j)(B_1^j + \beta B_2^j)$$

   Here α, β and $S(\alpha,\beta)$ each take values in ± 1.

4. We define the statistical average of the correlations between measurement outcomes:

$$\langle A_m B_n \rangle = \frac{1}{N} \sum_{j=1}^{N} \langle A_m^j B_n^j \rangle$$

   with $m, n \in \{1, 2\}$. From the bounds found in 3., write the bounds on $\left\langle \tilde{S} \right\rangle$ as a function of the correlations $\langle A_m B_n \rangle$

5. Choose the particular function $S(-1,-1) = -1$ and $S(\alpha,\beta) = 1$ otherwise. Write down the inequality satisfied by $< \tilde{S} >$ in the form

$$|\sum_{m,n=1}^{2} c_{mn} \langle A_m B_n \rangle | \leq 2$$

and give the coefficients c_{mn}. This is the CHSH inequality.

6. Design a classical strategy that may lead to violations of this inequality.

7. How many repetitions N (order of magnitude) should we perform to measure $\left\langle \tilde{S} \right\rangle$ with 1% error or less?

2 Experimental violation of CHSH inequality

We consider a source that outputs twin photons whose wavefunction is well approximated by:

$$|\psi >= \frac{1}{\sqrt{1+|\lambda|^2}} \left(\mathbb{1} + \frac{\lambda}{\sqrt{2}} \left(\hat{a}_v^\dagger \hat{b}_h^\dagger + \hat{a}_h^\dagger \hat{b}_v^\dagger \right) \right) |\text{vac}\rangle$$

where a, b refers to two different spatial modes, and h, v label two orthogonal polarizations.

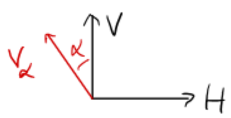
1. What nonlinear process is expected to produce such a biphoton state?

2. What are the requirements on the photon collection geometry to obtain such a state?

3. What is the probability of emitting a photon pair?

4. In the experiment we only keep results where both detectors on modes a and b count one photon (coincidence events). How can we simplify the wavefunction $|\psi >$ after this post-selection (now labeled $|\tilde{\psi}\rangle$)?
   You may also use the following ket notation for brievity:
   $$\hat{a}_v^\dagger \hat{b}_h^\dagger |\text{vac}\rangle = |v\rangle_a |h\rangle_b \quad \text{and} \quad \hat{a}_h^\dagger \hat{b}_v^\dagger |\text{vac}\rangle = |h\rangle_a |v\rangle_b$$

5. Alice and Bob collect photons in spatial modes a and b, respectively. They use linear polarisers transmitting vertical polarisations v_α, v_β rotated by angles α, β with respect to v. Write down the kets $|v_\alpha\rangle_a$ onto which these measurements project in the original basis.
   Hint: Use the classical expression.



6. Express the probability that both photons are transmitted through the polarizers as a function of α and β for the state $|\tilde{\psi} >$ of question 2.4.
   Hint: Start from Born's rule: $P_{\alpha,\beta} = \| \left(\langle v_\alpha | \otimes \langle v_\beta | \right) |\tilde{\psi}\rangle \|^2$

7. What joint transformations on α and β leave $P_{\alpha,\beta}$ unchanged?

8. To make the connection with part 1, we define the expectation value for joint measurements of polarization as:
   $$\langle A_\alpha B_\beta \rangle = P_{\alpha,\beta} + P_{\alpha_\perp,\beta_\perp} - P_{\alpha_\perp,\beta} - P_{\alpha,\beta_\perp}$$
   where $\alpha_\perp$ $\beta_\perp$ are angles perpendicular to α, β, respectively. In other words, this expectation value is 1, if the photons are always co-polarized in the rotated basis α, β and -1, if they are always cross polarized.
   Find two values α_1, α_2 and two values β_1, β_2 among the set $\left\{ -\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8} \right\}$ for which:
   $$| \langle A_{\alpha_1} B_{\beta_1} \rangle + \langle A_{\alpha_2} B_{\beta_1} \rangle + \langle A_{\alpha_2} B_{\beta_2} \rangle - \langle A_{\alpha_1} B_{\beta_2} \rangle | > 2$$

9. Such violation of CHSH inquality is indeed observed in the lab. What can you conclude about photons produced by SPDC? What class of probabilistic models describing the SPDC process must be excluded?