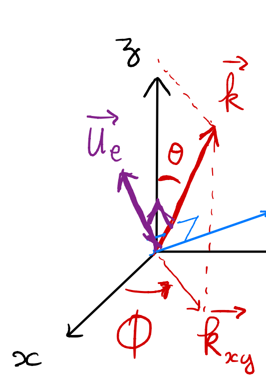
1) a)

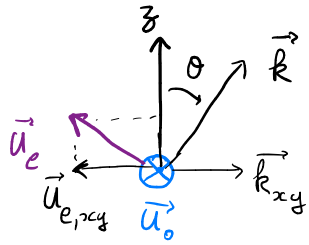


General geometry

* $\vec{u}_0 \in (x,y)$ plane and is perpendicular to $\vec{k}_{xy}$

* $\vec{u}_e$ belongs to $(\vec{k}, \vec{k}_{xy})$ plane and is $\perp$ to $\vec{k}$ and $\vec{u}_0$

* View along $\vec{u}_0$:



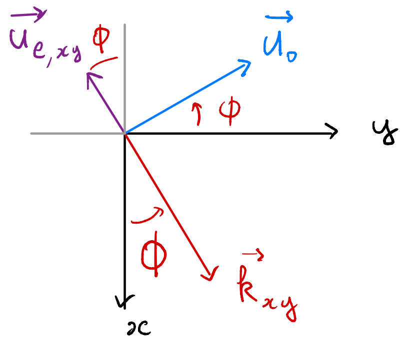
$\Rightarrow$ The projections on $\vec{u}_z$ are:

$$\vec{u}_0 \cdot \vec{u}_z = 0$$

$$\vec{u}_e \cdot \vec{u}_z = \sin\theta$$

Moreover the projection of $\vec{u}_e$ in the (x,y) plane has norm $\|\vec{u}_{e,xy}\| = \cos\theta$

* View from $\vec{u}_z$:



We see that :

$$\vec{u}_0 \cdot \vec{u}_x = -\sin\phi$$

$$\vec{u}_0 \cdot \vec{u}_y = \cos\phi$$

And

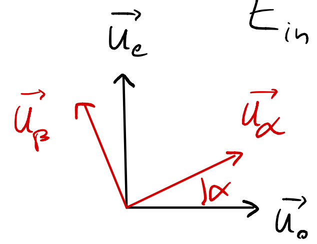
$$\vec{u}_{e,xy} \cdot \vec{u}_x = -\cos\phi$$

$$\vec{u}_{e,xy} \cdot \vec{u}_y = -\sin\phi$$

b) We therefore finally arrive at :

$$\vec{u}_0 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{u}_e = \begin{pmatrix} -\cos\theta\cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta \end{pmatrix}$$

c) $\vec{E}_{in}(\omega) = \frac{E_0}{2}\left[\cos\left(\frac{\gamma}{2}\right)\vec{u}_\alpha - i\sin\left(\frac{\gamma}{2}\right)\vec{u}_\beta\right]$



i. If $\gamma = 0, \pi \Rightarrow$ linear polarization with orientation controlled by α. $\vec{E} \parallel \vec{u}_o \Leftrightarrow \begin{cases} \gamma = 0 \text{ and } \alpha = 0 \\ \gamma = \pi \text{ and } \alpha = \frac{\pi}{2} \end{cases}$

ii. $\gamma = 0$ & $\alpha = \frac{\pi}{4}$ or $\gamma = \pi$ & $\alpha = \frac{3\pi}{4}$

iii. If $\gamma = \pm\frac{\pi}{2} \Rightarrow$ circular polarization

Remark: When the polarization is perfectly circular we expect that the angle α should not matter. Indeed, for $\gamma = \frac{\pi}{2}$:

$$A_o = \frac{E_0}{2\sqrt{2}} e^{i\alpha} \quad \text{and} \quad A_e = \frac{E_0}{2\sqrt{2}} \times -i\left(\cos\alpha + i\sin\alpha\right)$$

$$= \frac{E_0}{2\sqrt{2}} e^{-i\frac{\pi}{2}} e^{i\alpha}$$

↳ The common phase factor $e^{i\alpha}$ is irrelevant

d) We now want to write the wave as $\vec{E}_{in}(\omega) = A_o\vec{u}_o + A_e\vec{u}_e$:

(i) $\vec{E}_{in} \cdot \vec{u}_o = A_o = \frac{E_0}{2}\left(\cos\frac{\gamma}{2}\cos\alpha + i\sin\frac{\gamma}{2}\sin\alpha\right)$

(ii) $\vec{E}_{in} \cdot \vec{u}_e = A_e = \frac{E_0}{2}\left(\cos\frac{\gamma}{2}\sin\alpha - i\sin\frac{\gamma}{2}\cos\alpha\right)$

---

2) a) Neglecting walk-off means that we consider $\vec{E} \parallel \vec{D}$

At the input facet ($s = 0$) $\vec{E}(\omega, s=0) = A_o\vec{u}_o + A_e\vec{u}_e$

At any further position $s > 0$:

$$\vec{E}(\omega, z) = A_o\vec{u}_o e^{ik_o(\omega)s} + A_e\vec{u}_e e^{ik_e(\omega)s}$$

with $k_o(\omega) = \|\vec{k}_o(\omega)\| = n_o(\omega)\frac{\omega}{c}$

and $k_e(\omega) = \|\vec{k}_e(\omega)\| = n_\theta(\omega)\frac{\omega}{c}$ where $\frac{1}{n_\theta(\omega)^2} = \frac{\sin^2\theta}{n_e(\omega)^2} + \frac{\cos^2\theta}{n_o(\omega)^2}$

In a negative uniaxial crystal: $n_e(\omega) \leqslant n_\theta(\omega) \leqslant n_o(\omega)$

b) Quite generally $\vec{P}^{(2)}(2\omega) = \varepsilon_0 K \, \underline{\underline{\chi}}^{(2)}(-2\omega;\omega,\omega):\vec{E}(\omega)\vec{E}(\omega)$

i.e. $P_i^{(2)}(2\omega) = \varepsilon_0 K \chi^{(2)}_{ijk} E_j(\omega) E_k(\omega)$ with $K=\frac{1}{2}$ for SHG

(the $\vec{s}$ dependence of all fields is implicit above)

Injecting the expression from 2b) for $\vec{E}(\omega,\vec{s})$:

$$\vec{P}^{(2)}(2\omega) = \frac{1}{2}\varepsilon_0 \underline{\underline{\chi}}^{(2)} : \left(A_o \vec{u}_o e^{ik_o(\omega)s} + A_e \vec{u}_e e^{ik_e(\omega)s} \right)^2$$

$$= \frac{1}{2}\varepsilon_0 \underline{\underline{\chi}}^{(2)} : \Big(A_o e^{2ik_o(\omega)s} \vec{u}_o\vec{u}_o + A_e e^{2ik_e(\omega)s} \vec{u}_e\vec{u}_e + A_o A_e e^{i(k_o+k_e)s}(\vec{u}_o\vec{u}_e + \vec{u}_e\vec{u}_o) \Big)$$

We don't perform the contraction at this stage because $\chi^{(2)}_{ijk}$ is expressed in the crystal's x,y,z coordinates, not in the $\{\vec{u}_o,\vec{u}_e\}$ basis.

c) See lecture

d,e) • Growth of the ordinary wave at 2ω:

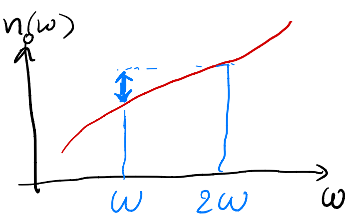
$$\frac{\partial A_o(2\omega)}{\partial s} \propto \vec{u}_o \cdot \vec{P}^{(2)}(2\omega)\, e^{-ik_o(2\omega)s}$$

$$= \frac{\varepsilon_0}{2}\Big\{ \vec{u}_o\cdot\chi^{(2)}:\vec{u}_o\vec{u}_o A_o^2 e^{i\Delta k_{ooo}s} + \vec{u}_o\cdot\chi^{(2)}:\vec{u}_e\vec{u}_e A_e^2 e^{i\Delta k_{oee}s} + \vec{u}_o\cdot\chi^{(2)}:(\vec{u}_o\vec{u}_e+\vec{u}_e\vec{u}_o)A_oA_e e^{i\Delta k_{oeo}s} \Big\}$$

with: (i) $\Delta k_{ooo} = 2k_o(\omega) - k_o(2\omega) = \frac{\omega}{c}\left(2n_o(\omega) - 2n_o(2\omega)\right)$

$$= \frac{2\omega}{c}\left[n_o(\omega) - n_o(2\omega)\right] < 0$$

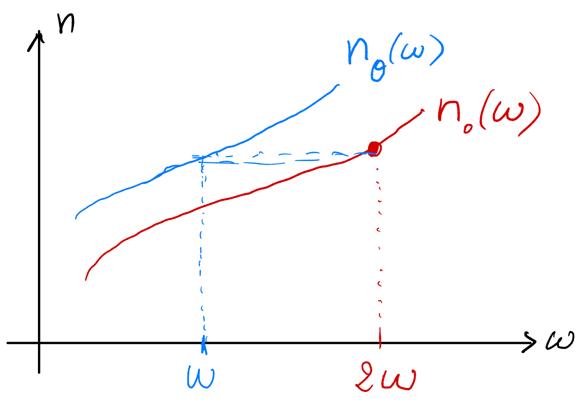
↳ This term is never zero due to dispersion



(ii) $\Delta k_{oee} = \frac{2\omega}{c}\left(n_\theta(\omega) - n_o(2\omega)\right)$

↳ Can be zero in positive uniaxial crystals where $n_\theta > n_o$

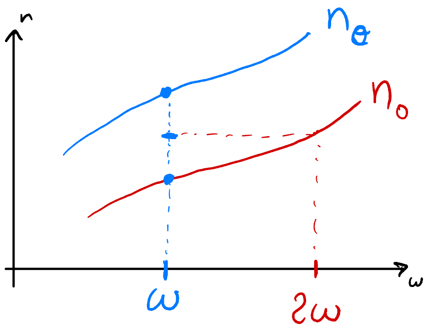
(Type I phase matching)



(iii) $\Delta k_{oeo} = \frac{\omega}{c}\left(n_\theta(\omega) + n_o(\omega) - 2n_o(2\omega)\right)$

↳ $\Delta k_{oeo} = 0 \iff n_o(2\omega) = \frac{n_\theta(\omega) + n_o(\omega)}{2}$

⇒ Type II phase matching in positive crystals

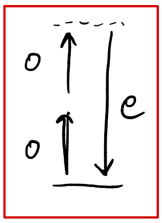


- Growth of the extraordinary wave at 2ω

$$\frac{\partial A_e(2\omega)}{\partial s} \propto \vec{u}_e \cdot \vec{P}^{(2)}(2\omega)\, e^{-ik_e(2\omega)s}$$

$$= \frac{\varepsilon_0}{2}\left\{ \vec{u}_e \cdot \chi^{(2)} : \vec{u}_o \vec{u}_o A_o^2 e^{i\Delta k_{eoo} s} + \vec{u}_e \cdot \chi^{(2)} : \vec{u}_e \vec{u}_e A_e^2 e^{i\Delta k_{eee} s} + \vec{u}_e \cdot \chi^{(2)} : (\vec{u}_e \vec{u}_o + \vec{u}_o \vec{u}_e) A_o A_e e^{i\Delta k_{eeo} s} \right\}$$

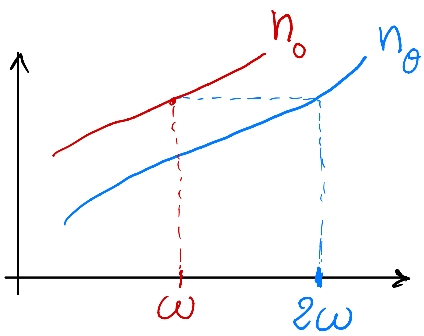
with:



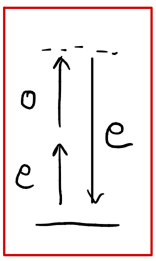
(i) $\Delta k_{eoo} = \frac{2\omega}{c}\left(n_o(\omega) - n_\theta(2\omega)\right)$

$\Delta k_{eoo} = 0 \iff n_\theta(2\omega) = n_o(\omega)$

↳ Type I phase matching in negative crystal



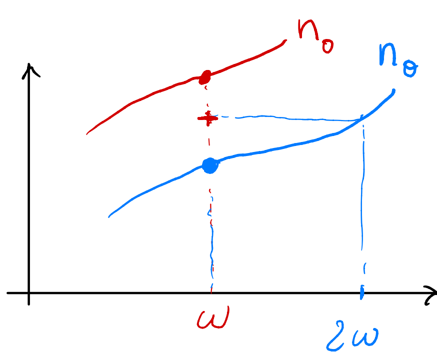
(ii) $\Delta k_{eee} = \frac{2\omega}{c}\left(n_\theta(\omega) - n_\theta(2\omega)\right)$ → never zero.



(iii) $\Delta k_{eeo} = \frac{\omega}{c}\left(n_\theta(\omega) + n_o(\omega) - 2n_\theta(\omega)\right)$

$\Delta k_{eeo} = 0 \iff n_\theta(2\omega) = \frac{n_\theta(\omega) + n_o(\omega)}{2}$

↳ Type II phase matching in negative crystal.

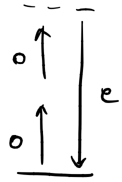


---

3) a) The angle θ is set by phase matching

We can still optimize Φ, α, γ.

b) * <u>Type I SHG in KDP</u> corresponds to :



$$P_e^{(2)}(2\omega) \propto \vec{u_e} \cdot \chi^{(2)} : \vec{u}_o \vec{u}_o A_o^2$$

We must express $\vec{u}_o$ in x, y, z and then we can use the contracted d matrix notation to find $\chi^{(2)} : \vec{u}_o \vec{u}_o$

$$\vec{u}_o = -\sin\phi \, \vec{u}_x + \cos\phi \, \vec{u}_y$$

$\hookrightarrow$ $\vec{u}_o \vec{u}_o = \sin^2\phi \, \vec{u}_x \vec{u}_x - \sin\phi\cos\phi \left(\vec{u}_x \vec{u}_y + \vec{u}_y \vec{u}_x\right) + \cos^2\phi \, \vec{u}_y \vec{u}_y$

Thus, in the x, y, z basis the components of $\vec{P}^{(2)}(2\omega)$ are :

$$\varepsilon_0 A_0^2 \begin{bmatrix} d_{11} & \cdots & d_{16} \\ d_{22} & \cdots & d_{26} \\ d_{33} & \cdots & d_{36} \end{bmatrix} \times \begin{bmatrix} \sin^2\phi \\ \cos^2\phi \\ 0 \\ 0 \\ 0 \\ -\sin\phi\cos\phi \end{bmatrix} = -\varepsilon_0 A_0^2 \, d_{36} \sin\phi\cos\phi \, \vec{u_z}$$

because only $d_{14}, d_{25}, d_{36} \neq 0$

We now project $\vec{P}^{(2)}$ on $\vec{u_e}$:

$$\vec{P}^{(2)}(2\omega) \cdot \vec{u_e} = -\varepsilon_0 A_0^2 \, d_{36} \sin\phi\cos\phi \, \vec{u_z} \cdot \vec{u_e}$$

$$= -\varepsilon_0 A_0^2 \boxed{d_{36} \, \frac{1}{2} \sin 2\phi \, \sin\theta} \longrightarrow d_{eff}$$

c) To optimize SHG, we must therefore maximize $\| A_o^2 \sin 2\phi \|$ $(A_o \in \mathbb{C})$

o $\longrightarrow$ choose $\boxed{\phi = \frac{\pi}{4}}$; i.e. $\boxed{\vec{k} \cdot \vec{u_x} = \vec{k} \cdot \vec{u_y}}$

o $\longrightarrow$ Maximize $\|A_o^2\|^2 = A_o^2 \times A_o^{*2} = (A_o A_o^*)^2 = \left(\cos^2\frac{\gamma}{2}\cos^2\alpha + \sin^2\frac{\gamma}{2}\sin^2\alpha\right)^2 \frac{E_o^4}{16}$

Two possibilities :

(i) $\gamma = \alpha = 0$ (first term) $\rightarrow$ <u>linear polarisation along $\vec{u_o}$</u>

(ii) $\gamma = \pi$; $\alpha = \frac{\pi}{2}$ (second term) $\rightarrow$ <u>linear polarisation along $\vec{u_o}$</u>

$\hookrightarrow$ consistent with Type I SHG.

d) * Type II SHG in KDP : $P_e^{(2)}(2\omega) = \frac{1}{2}\varepsilon_0 \vec{u}_e \cdot \chi^{(2)} : (\vec{u}_o\vec{u}_e + \vec{u}_e\vec{u}_o) A_o A_e$

$\vec{u}_o = -\sin\phi \, \vec{u}_x + \cos\phi \, \vec{u}_y$

$\vec{u}_e = -\cos\theta\cos\phi \, \vec{u}_x - \cos\theta\sin\phi \, \vec{u}_y + \sin\theta \, \vec{u}_z$

$\hookrightarrow \vec{u}_o\vec{u}_e = \cos\theta \, \frac{1}{2}\sin 2\phi \, (\vec{u}_x\vec{u}_x - \vec{u}_y\vec{u}_y) - \cos\theta\cos^2\phi \, \vec{u}_y\vec{u}_x$
$+ \cos\theta\sin^2\phi \, \vec{u}_x\vec{u}_y - \sin\phi\sin\theta \, \vec{u}_x\vec{u}_z + \cos\phi\sin\theta \, \vec{u}_y\vec{u}_z$

$\vec{u}_e\vec{u}_o = \dots$

Instead of writing down all terms, we can use the fact that only d_{14}, $d_{25} = d_{14}$ and d_{36} are $\neq 0$; so we only need to find the terms in $\vec{u}_y\vec{u}_z + \vec{u}_z\vec{u}_y$, $u_x u_z + u_z u_x$ and $\vec{u}_x\vec{u}_y + \vec{u}_y\vec{u}_x$, which are: $\vec{u}_o\vec{u}_e + \vec{u}_e\vec{u}_o = (\dots) + \cos\theta(\sin^2\phi - \cos^2\phi)(\vec{u}_x\vec{u}_y + \vec{u}_y\vec{u}_x)$

$$- \sin\theta\sin\phi \, (\vec{u}_x\vec{u}_z + \vec{u}_z\vec{u}_x)$$
$$+ \sin\theta\cos\phi \, (\vec{u}_y\vec{u}_z + \vec{u}_z\vec{u}_y)$$

Therefore:
$$P_e^{(2)}(2\omega) = \varepsilon_0 A_o A_e \Big\{ d_{14}\big(\sin\theta\cos\theta(\sin^2\phi - \cos^2\phi)\big) + d_{36}\sin\theta\cos\theta(\sin^2\phi - \cos^2\phi) \Big\}$$
$$= -\frac{\varepsilon_0}{2} A_o A_e \sin 2\theta \cos 2\phi \, (d_{14} + d_{36})$$

$\hookrightarrow$ The optimal angles for $\vec{k}_{xy}$ are $\phi = 0$ or $\frac{\pi}{2}$; i.e. $/\!/ \vec{u}_x$ or $/\!/ \vec{u}_y$

Then, we look for the optimal polarization by maximizing:

$$\|A_o A_e\|^2 = (A_o A_e) \times (A_o A_e)^* = \frac{E_0^4}{16}\left[\left(\sin\alpha\cos\alpha\left(\cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2}\right) - i\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}(\cos^2\alpha - \sin^2\alpha)\right) \times c.c.\right]$$

$$= \frac{E_0^4}{16}\left[\left(\frac{1}{2}\sin 2\alpha - i\,\frac{1}{2}\sin\gamma\cos 2\alpha\right) \times c.c.\right]$$

$$= \frac{E_0}{16} \times \frac{1}{4}\left(\sin^2 2\alpha + \sin^2\gamma\cos^2 2\alpha\right)$$

(i) $\alpha = \frac{\pi}{4}$ or $\frac{3\pi}{4}$; γ = whatever $\rightarrow$ diagonal or any elliptical polarisation

(ii) $\alpha = 0$ or $\frac{\pi}{2}$; $\gamma = \frac{\pi}{2}$ $\rightarrow$ circular polarisation.