Problem set 07

Nonlinear Optics for Quantum Technologies

April 3, 2025

1 Optimization of second harmonic generation in a uniaxial crystal

1.1 Geometry of the experiment

We consider a uniaxial crystal (refractive indices n_o, n_e) pumped by a monochromatic plane wave of frequency ω. The wavevector $\boldsymbol{k}(\omega)$ makes an angle θ with the optic axis z, and the projection of $\boldsymbol{k}(\omega)$ onto the x, y plane makes an angle ϕ with the x axis.

a) Draw $\boldsymbol{k}(\omega)$ and the two normal mode polarization unit vectors $\boldsymbol{u}_o$ (ordinary) and $\boldsymbol{u}_e$ (extraordinary) in the x, y, z coordinate system. We choose the orientations of $\{\boldsymbol{k}(\omega), \boldsymbol{u}_o, \boldsymbol{u}_e\}$ to form a right-handed triad.

b) Express the x, y, z components of the unit vectors $\boldsymbol{u}_o$ and $\boldsymbol{u}_e$ as a function of θ and ϕ.

c) Just at the input facet of the crystal, we write the polarization state of the pump beam

$$\boldsymbol{E}_{in}(\omega) = \frac{E_0}{2}\left[\cos\left(\frac{\gamma}{2}\right)\boldsymbol{u}_\alpha - i\sin\left(\frac{\gamma}{2}\right)\boldsymbol{u}_\beta\right]$$

where $\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta$ are two orthogonal unit vectors in the $(\boldsymbol{u}_o, \boldsymbol{u}_e)$ plane, forming an angle α between $\boldsymbol{u}_o$ and $\boldsymbol{u}_\alpha$ (and between $\boldsymbol{u}_e$ and $\boldsymbol{u}_\beta$). What are the values of (α, γ) corresponding to the following polarization states:

i. linear along $\boldsymbol{u}_o$, linear along $\boldsymbol{u}_e$
ii. linear diagonal (at 45° between $\boldsymbol{u}_o$ and $\boldsymbol{u}_e$)
iii. circular

d) In the following, we neglect beam walk-off by considering $\boldsymbol{E} \parallel \boldsymbol{D}$. Decompose $\boldsymbol{E}_{in}$ into the normal modes of polarization as

$$\boldsymbol{E}_{in} = A_o(\omega)\boldsymbol{u}_o + A_e(\omega)\boldsymbol{u}_e$$

(give A_o and A_e as a function of α and γ).

1.2 Nonlinear polarization and second harmonic generation

a) We note $s = \boldsymbol{r} \cdot \frac{\boldsymbol{k}}{k}$ the propagation distance along the pump beam direction. Express the pump wave $\boldsymbol{E}(\omega, s)$ for any $s > 0$ as a function of $\boldsymbol{k}_o(\omega)$ and $\boldsymbol{k}_e(\omega)$, considering that the amplitudes A_o, A_e remain constant (undepleted pump approximation). How are $k_o(\omega)$ and $k_e(\omega)$ to related $\frac{\omega}{c}$, n_o and n_e?

b) Give the expression of the nonlinear polarization $\boldsymbol{P}^{(2)}(2\omega)$ at the second harmonic frequency for the general pump beam in 1.1.d). Why cannot we perform the tensor contraction at this stage? You may use the basis-independent notation $\boldsymbol{a} \cdot \chi^{(2)} : \boldsymbol{b}\,\boldsymbol{c}$.

c) From Maxwell's equations, neglecting beam walk-off (i.e., making the approximation $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$) and assuming a slowly varying envelope, show that

$$\frac{\partial}{\partial s}A_o(2\omega) \propto \boldsymbol{u}_o \cdot \boldsymbol{P}^{(2)}(2\omega)e^{-i\,k_o(2\omega)\,s}$$

d) Expand the right-hand term with the expression of $\boldsymbol{P}^{(2)}(2\omega)$ found in a) and identify three independent terms proportional to A_o^2, A_e^2 and $A_o A_e$, having distinct phase mismatch values Δk_{ooo}, Δk_{oee} and Δk_{oeo}, respectively. The subscripts ooo, oee, oeo stand for type 0, type I and type II phase matching, respectively. What is their meaning in terms of annihilated and created photon polarization?

e) Which term can never be phase-matched? Explain how to phase-match the other two terms. Should the crystal be positive ($n_e > n_0$) or negative ($n_e < n_0$)?

f) Repeat questions d) and e) for the growth of the extra-ordinary wave amplitude $A_e(2\omega)$. What are the photon polarizations for type 0, type I and type II phase matching in this case?

1.3 Type I phase matching in KDP

We now consider the negative uniaxial crystal KDP (Monopotassium phosphate, point group -4 2 m) under type I phase matching, and we want to find the optimal parameters $\theta, \phi, \alpha, \gamma$ for efficient SHG.

a) What parameter(s) is/are set by the phase matching condition? We now assume that $\Delta k_{eoo} = 0$.

b) Using the decomposition of $\boldsymbol{u}_o$ and $\boldsymbol{u}_e$ in the x, y, z basis of the crystal (Sec. 1.1) and the contracted notation d_{il} compute the expression of $d_{\text{eff}} = \boldsymbol{u}_e \cdot \underline{\underline{d}} : \boldsymbol{u}_o \, \boldsymbol{u}_o$.

   Reminder: for the point group -4 2 m the only non-zero tensor elements are $d_{14} = d_{25}$ and d_{36}.

c) Deduce what parameters must be optimized and what are their optimal values to maximize the SHG intensity.

d) (*optional*) Repeat the same analysis for type II phase matching in KDP.