Problem Set 06

1 – Optical Parametric Amplifier (OPA)

1-a) We neglect beam walk-off : $\vec{\nabla}.\vec{E} = 0$

We expand $\vec{P}(\vec{E}) = \vec{P}^{(1)}(\vec{E}) + \vec{P}^{(2)}(\vec{E}) + \dots$

Then Maxwell equations can be written as :

$$-\vec{\nabla}^2\vec{E}(z,t) + \frac{1}{c^2}\ddot{\vec{E}}(z,t) + \frac{1}{\varepsilon_0 c^2}\ddot{\vec{P}}^{(1)}(z,t) = -\frac{1}{\varepsilon_0 c^2}\ddot{\vec{P}}^{(n)}(z,t)$$

We Fourier transform this expression : for each ω_α, we get

(1) $$\nabla^2\vec{E}_\alpha(z) + \frac{\omega_\alpha^2(1+\chi_\alpha^{(1)})}{c^2}\vec{E}_\alpha(z) = -\frac{\omega_\alpha^2}{\varepsilon_0 c^2}\vec{P}_\alpha^{(n)}(z)$$

(we recall that $\vec{E}_\alpha(z) = A_\alpha(z)\, e^{jk_\alpha z}\vec{u}_\alpha$,

$$\vec{P}_\alpha^{(1)}(z) = \varepsilon_0\chi_\alpha^{(1)}\vec{E}_\alpha(z)$$

and $k_\alpha = \frac{n_\alpha\omega_\alpha}{c}$, i.e. $k_\alpha^2 = \frac{\omega_\alpha^2}{c^2}(1+\chi_\alpha^{(1)})$)

Using the slowly varying envelope approximation, $\frac{\partial^2 A_\alpha}{\partial z^2} \ll k_\alpha\frac{\partial A_\alpha}{\partial z}$

we get : $$\nabla^2\vec{E}_\alpha(z) = \left(2j\, k_\alpha\frac{\partial A_\alpha}{\partial z} - k_\alpha^2 A_\alpha(z)\right)e^{ik_\alpha z}\vec{u}_\alpha$$

Injecting in (1) gives (and projecting along $\vec{u}_\alpha$)

(2) $$\boxed{\frac{\partial A_\alpha}{\partial z}(z) = \frac{j\,\omega_\alpha}{2\,\varepsilon_0 c n_\alpha}\vec{P}_\alpha^{(n)}(z)\,.\,\vec{u}_\alpha\, e^{-jk_\alpha z}}$$

The material we consider is a second-order non-linear crystal : we must consider $\vec{P}_\alpha^{(2)}$. Moreover, we recall that $\omega_p = \omega_i + \omega_s$

(i) For $\alpha = p$: $\vec{P}_p^{(2)}(z) = \varepsilon_0 K\chi^{(2)}(-\omega_p : \omega_i, \omega_s)\vec{E}_i(z)\vec{E}_s(z)$

Thus $$\frac{\partial A_p}{\partial z}(z) = \frac{j\omega_p}{2\varepsilon_0 c\, n(\omega_p)}\varepsilon_0\,\vec{u}_p . K\left(\chi^{(2)}(-\omega_p : \omega_i,\omega_s) : \vec{u}_i.\vec{u}_s\right)A_i(z)A_s(z)e^{j\Delta k z}$$

with $\Delta k = k_i + k_s - k_p$

(ii) Similarly, for $\alpha = i$: $\vec{P}_i^{(2)}(z) = \varepsilon_0 K\chi^{(2)}(-\omega_i : \omega_p, -\omega_s)\vec{E}_p(z)\vec{E}_s^*(z)$

(iii) and $\alpha = s$: $\vec{P}_s^{(2)}(z) = \varepsilon_0 K\chi^{(2)}(-\omega_s ; \omega_p, -\omega_i)\vec{E}_p(z)\vec{E}_i^*(z)$

(note that in the case of DFG, $K = 1$)

According to full permutation symmetry,

$$\chi^{(2)}_{pis}(-\omega_p;\omega_i,\omega_s) = \chi^{(2)*}_{ips}(-\omega_i:\omega_p,-\omega_s) = \chi^{(2)*}_{spi}(-\omega_s;\omega_p,-\omega_i)$$

Finally, we end up with:

$$\left|\begin{aligned}
\frac{\partial A_p}{\partial z}(z) &= j\frac{\omega_p}{2cn(\omega_p)}\chi^{(2)*}_{eff}\,A_i(z)\,A_s(z)\,e^{j\Delta kz}\\
\frac{\partial A_s}{\partial z}(z) &= j\frac{\omega_s}{2cn(\omega_s)}\chi^{(2)}_{eff}\,A_p(z)\,A_i^*(z)\,e^{-j\Delta kz}\\
\frac{\partial A_i}{\partial z}(z) &= \frac{j\omega_i}{2cn(\omega_i)}\chi^{(2)}_{eff}\,A_p(z)\,A_s^*(z)\,e^{-j\Delta kz}
\end{aligned}\right.$$

with $\Delta k = k_i + k_s - k_p$ and $\chi^{(2)}_{eff} = \chi^{(2)*}_{pis}(-\omega_p;\omega_i,\omega_s)$

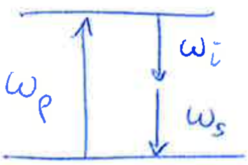
b) The equations with reduced variables are:

$$(3)\quad\left|\begin{aligned}
\frac{\partial a_p}{\partial z}(z) &= j\,\xi^*\,a_i(z)\,a_s(z)\,e^{j\Delta kz}\\
\frac{\partial a_i}{\partial z}(z) &= j\,\xi\,a_p(z)\,a_s^*(z)\,e^{-j\Delta kz}\\
\frac{\partial a_s}{\partial z}(z) &= j\,\xi\,a_p(z)\,a_i^*(z)\,e^{-j\Delta kz}
\end{aligned}\right.$$

c) We recall that light intensity in a medium, in W/m^2, is $I = \frac{1}{2}\varepsilon_0 c n E^2$ and the photon energy is $\hbar\omega$, in J.
ϕ_m has the unit of $|a_m|^2$, then it has dimension $\frac{W}{J}\cdot m^{-2} = m^{-2}\cdot s^{-1}$.
And based on the above observations, it can be interpreted as the ~~~~ number of photons per unit of time and area, i.e the photon flux per unit area in the beam.

d) $$\frac{d\phi_m}{dz} = a_m^*\frac{da_m}{dz} + a_m\frac{da_m^*}{dz}$$

From (b) we get $\boxed{\frac{d\phi_s}{dz} = \frac{d\phi_i}{dz} = -\frac{d\phi_p}{dz}}\left(= j\,\xi^*\,a_p^*\,a_i\,a_s\,e^{j\Delta kz} + c.c.\right)$



In DFG, the annihilation of a pump photon results in the creation of a signal photon and an idler photon (and vice versa).

e) The total power is $P = \sum_m \phi_m \hbar \omega_m$

and $\frac{dP}{dz} = \sum_m \frac{d\phi_m}{dz} \hbar \omega_m$.

With the Manley-Rowe relations, and since $\omega_p = \omega_i + \omega_s$

we find $\frac{dP}{dz} = 0$: the total power is conserved.

2. Phase matching and phase mismatch

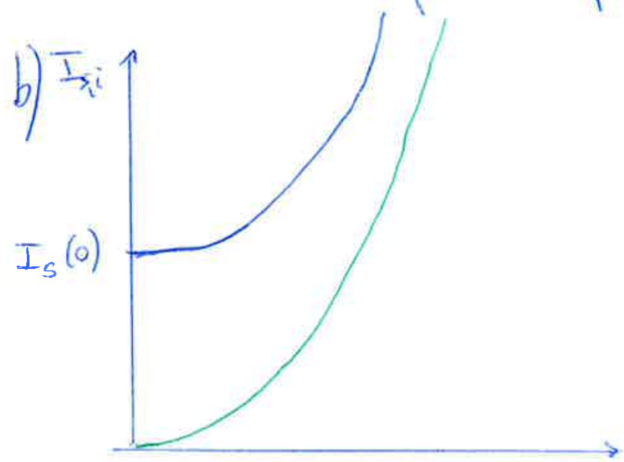
a) The equations for the evolution of $a_{i,s}(z)$ are:

$$\boxed{\frac{d^2}{dz^2} a_{i,s} = |\xi a_p|^2 a_{i,s}}$$

The general solutions are: $a_{i,s}(z) = a_{i,s}(0) \cosh gz + j\, a^*_{s,i}(0) \sinh gz$

where $g = |\xi a_p|$ (we can also chose ξa_p to be real)

g has the dimension of m^{-1}, it can be maximized by increasing the drive amplitude a_p, i.e the pumping power.

b) 

For $|A_i(z=0)| = 0$,

then $I_s(z) = \hbar\omega_s\, a_s^*(z)\, a_s(z)$

$= \hbar\omega_s |a_s(0)|^2 \cosh^2 gz$

$= I_s(0) \cosh^2 gz$

and $I_i(z) = \frac{\omega_i}{\omega_s} I_s(0) \sinh^2 gz$

Amplification factor: $\cosh^2 gz$

Limits: (i) $gz \ll 1$: $I_s(z) \approx I_s(0) \times (1 + (gz)^2) + O((gz)^4))$

$I_i(z) \approx \frac{\omega_i}{\omega_s} I_s(0) \times gz^2 + O((gz)^4)$

(ii) $gz \gg 1$: $I_s(z) \sim I_s(0) \frac{e^{2gz}}{4}$

$I_i(z) \sim \frac{\omega_i}{\omega_s} I_s(0) \frac{e^{2gz}}{4}$

If $|A_s(0)| = 0$, then $I_s(z) = I_i(z) = 0$. At least one beam must have non-zero amplitude for DFG process to take place.

c) Negative uniaxial crystal : $\varepsilon = \varepsilon_0 \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}$, with $n_e < n_o$

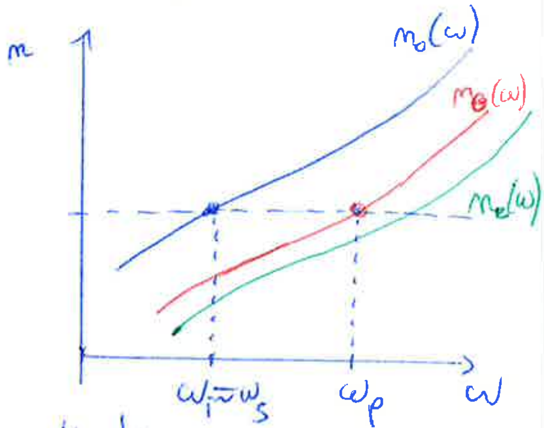
Phase matching condition : $\Delta k = 0$

$$\Leftrightarrow \quad n(\omega_i)\,\omega_i + n(\omega_s)\,\omega_s = n(\omega_p)\,\omega_p$$

- Type I phase matching : $\vec{u}_i = \vec{u}_s$

Assuming $\omega_i \approx \omega_s \approx \frac{\omega_p}{2}$

the condition becomes $n(\omega_p) \approx \frac{1}{2}\left(n(\omega_i) + n(\omega_s)\right)$



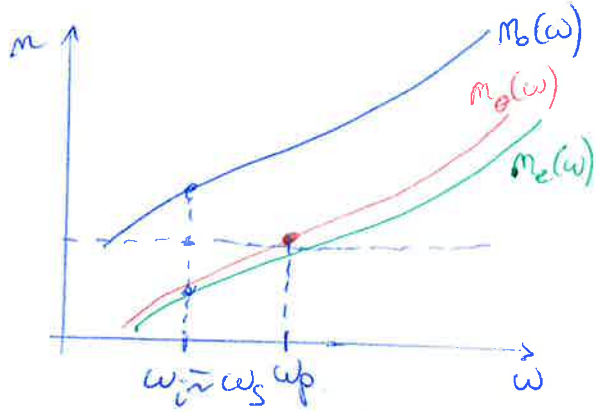
since $\vec{u}_i = \vec{u}_s$, then $n(\omega_i) \approx n(\omega_s)$

the polarization of the pump must be set at an angle θ from the signal and idler such that $\quad n_\theta(\omega_p) \approx n_o(\omega_i)$

the signal and idler should be along the an ordinary direction

(alternatively, the pump can be set along the extraordinary direction and the angle θ of signal and idler chosen appropriately)

- Type II phase matching : $\vec{u}_i \perp \vec{u}_s$



The signal should be along an ordinary direction and idler along extraordinary, and the pump polarization should be chosen such that

$$n_\theta(\omega_p) \approx \frac{1}{2}\left(n_o(\omega_s) + n_e(\omega_i)\right)$$

d) Using $\frac{\partial \alpha_{s,i}}{\partial z}(z) = \left(\frac{\partial a_{s,i}}{\partial z}(z) + j\frac{\Delta k}{2}\, a_{s,i}(z)\right) e^{j\frac{\Delta k}{2}z}$ and injecting in (3)

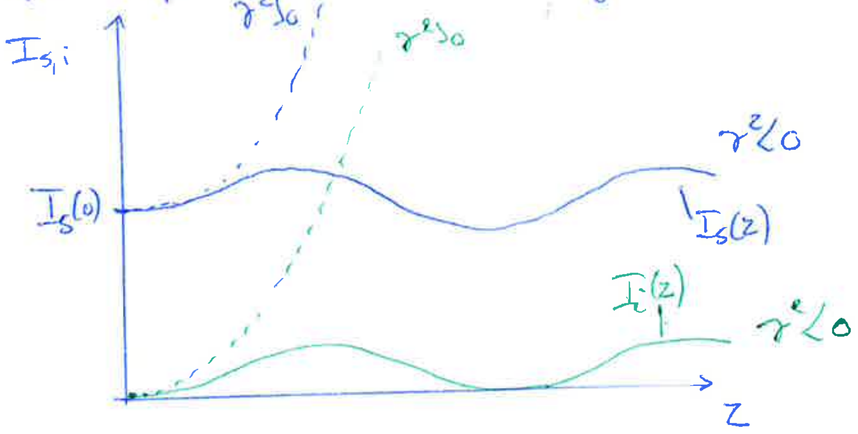
we get $\frac{\partial^2 \alpha_{s,i}}{\partial z^2}(z) = \gamma^2 \alpha_{s,i}(z) \quad$ with $\gamma^2 = |g|^2 - \frac{\Delta k^2}{4}$

e) From the equation above we see that a small phase mismatch will lead to a reduced amplification factor. However this can be compensated by increasing the pump power (i.e. increasing $|g|$).

Note: if $\gamma^2 = |g|^2 - \frac{\Delta k^2}{4} < 0$, the solutions are of the form

$$\alpha_{s,i}(z) = B_{s,i} \cos|\gamma|z + C_{s,i} \sin|\gamma|z$$

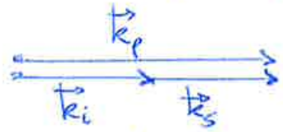
i.e the ~~start~~ field amplitudes are oscillating and there is no amplification overall.



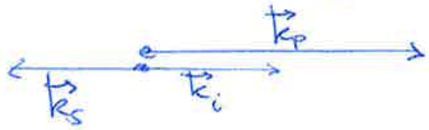
The pump power can still be increased to reach $\gamma^2 > 0$ and recover some amplification.

3. Optical parametric oscillator (OPO)

a) Forward direction :



Backward direction :



In the backward direction, the phase mismatch becomes $\Delta k = -k_s + k_i - k_p \neq 0$. Thus, we are far away from phase matching : there is no amplification. $(\Delta k = -2k_s)$

b) In one round-trip,

amplification : $I_s(L) = I_s(0) \cosh^2 gL$

losses : $I_s'(0) = (1-\eta_s) I_s(L)$

Gain factor per round-trip $G = (1-\eta_s)\cosh^2 gL$

The threshold condition is $G > 1$

c) For $\eta_s, gL \ll 1$, the threshold condition is $(1+(gL)^2) > \frac{1}{1-\eta_s}$

$\Leftrightarrow \boxed{(gL)^2 > \eta_s}$

d) In the limit of small gain and loss, the evolution of photon flux per round-trip due to losses is:

$$\Delta\phi_{s,i}^{(loss)} = -\eta_{s,i}\,\phi_{s,i}$$

Since $\phi_m = a_m^* a_m$, we deduce that the losses for the amplitudes are:

$$\Delta a_{s,i}^{(loss)} = -\frac{\eta_{s,i}}{2}\,a_{s,i}$$

For small gain, the evolution of amplitudes due to the nonlinear process are:

$$\Delta a_{s,i}^{(gain)} = jgL a_{i,s}^*$$

Eventually, the variations per round-trip are:

$$\begin{cases} \Delta a_s = -\frac{\eta_s}{2} a_s + jgL a_i^* \\ \Delta a_i = -\frac{\eta_i}{2} a_i + jgL a_s^* \end{cases}$$

e) From the Manley-Rowe relations, we know that photons in the signal and idler beams are created in pairs. In order to achieve double amplification, we thus need that the gain compensate losses for each channel, i.e. $\Delta a_i = \Delta a_s = 0$.

f) We can re-write the variation per round-trip as:

$$\Delta a_s = -\frac{\eta_s}{2} a_s + jgL\,a_i^*$$
$$\Delta a_i^* = -jgL\,a_s - \frac{\eta_i}{2}\,a_i^*$$

Non-trivial solutions for $\Delta a_s = \Delta a_i^* = 0$ exist if the determinant of the matrix $\begin{pmatrix} -\frac{\eta_s}{2} & jgL \\ -jgL & -\frac{\eta_i}{2} \end{pmatrix}$ is zero,

i.e $\frac{\eta_s\eta_i}{4} - (gL)^2 = 0$

The new threshold condition is $\boxed{(gL)^2 = \frac{\eta_s\eta_i}{4}}$

g) Singly resonant threshold: $(gL)^2 > \eta_s$
Doubly resonant threshold: $(gL)^2 > \frac{\eta_s \eta_i}{4}$

Thus, the doubly resonant case allows for much smaller pump power needed to reach threshold (the threshold condition is reduced by a factor $\frac{\eta_i}{4}$, which can be smaller than a few percent)
However, in practice it can be difficult to fabricate a cavity that is resonant for both ω_i and ω_s, so in most cases a singly-resonant OPO is preferred.