# Week 06 – problem set (assignment)

Nonlinear Optics for Quantum Technologies

March 27, 2025

## 1 Difference frequency generation

An optical parametric amplifier (OPA) exploits difference frequency generation (DFG) in a second-order non-linear crystal to amplify a signal beam at frequency $\omega_s$ using a strong external pump beam at frequency $\omega_p$, and generating an idler beam at frequency $\omega_i = \omega_p - \omega_s$ in the process. We consider that all beams are linearly polarized monochromatic plane waves:

$$\vec{E}_m(z,t) = \frac{1}{2}(A_m(z)e^{j(k_m z - \omega_m t)}\vec{u}_m + c.c.) \tag{1}$$

with $m = p, s, i$. (We use $j = \sqrt{-1}$ to avoid confusion with the index $i$ for idler.) Assuming a fixed set of polarization directions $\vec{u}_{p,s,i}$, the second order nonlinear tensor can be replaced by it's effective value, denoted by $\chi^{(2)}_{\text{eff}}$.

### 1.1 Coupled propagation equations and Manley-Rowe relations

a) Under the slowly varying envelope approximation and neglecting walk-off, show that the interaction between the three waves is given by the following system of coupled differential equations:

$$\frac{\partial A_p(z)}{\partial z} = j\frac{\omega_p}{2c\,n(\omega_p)}\chi^{(2)}_{\text{eff}} A_i(z)A_s(z)e^{j\Delta k z} \tag{2}$$

$$\frac{\partial A_s(z)}{\partial z} = j\frac{\omega_s}{2c\,n(\omega_s)}\chi^{(2)}_{\text{eff}} A_p(z)A_i^*(z)e^{-j\Delta k z} \tag{3}$$

$$\frac{\partial A_i(z)}{\partial z} = j\frac{\omega_i}{2c\,n(\omega_i)}\chi^{(2)}_{\text{eff}} A_p(z)A_s^*(z)e^{-j\Delta k z} \tag{4}$$

Give the expression of $\Delta k$ and $\chi^{(2)}_{\text{eff}}$.

b) We define the reduced variables

$$a_m(z) = \sqrt{\frac{n_m\, c\, \varepsilon_0}{2\hbar\omega_m}}A_m(z)$$

where $m = p, s, i$ and $n_m = n(\omega_m)$. Rewrite the system of coupled equations in terms of the reduced variables. To lighten the notation, you may introduce the quantity $\xi = \sqrt{\frac{\hbar\omega_p\omega_s\omega_i}{2n_p n_s n_i \varepsilon_0 c^2}}\chi^{(2)}_{\text{eff}}$.

c) What is the dimension of $\phi_m(z) = a_m^* a_m$? Rewrite it to let the beam intensity appear explicitly and justify thereby that it can be interpreted as the photon flux in the beam at frequency $\omega_m$.

d) From the coupled propagation equations, show that $\frac{\mathrm{d}\phi_s}{\mathrm{d}z} = \frac{\mathrm{d}\phi_i}{\mathrm{d}z} = -\frac{\mathrm{d}\phi_p}{\mathrm{d}z}$. How do you interpret this result in terms of photon annihilation or creation in the three-wave mixing process? This is called the **Manley-Rowe relation**.

e) Express the total power in the three beams and compute its derivative with respect to $z$. How is the total power changing?

## 1.2 Phase matching and phase mismatch

a) We first assume **perfect phase matching** ($\Delta k = 0$) and neglect pump depletion ($a_p$ = constant). Calculate the evolution of signal and idler beam amplitudes along the propagation direction. Express the solutions in terms of the initial amplitudes $a_m(z=0)$ and the functions $\cosh(gz)$ and $\sinh(gz)$. What is the expression of the gain coefficient $g$? What is it's dimension? How to maximize it?

b) Sketch the evolution of intensities, $I_s(z)$ and $I_i(z)$, in the signal and idler beams along the propagation direction $z$, assuming that $|A_s(z=0)| > 0$ and $|A_i(z=0)| = 0$. Give their expressions in the limits (i) $z \ll g^{-1}$ and (ii) $z \gg g^{-1}$.

   For an arbitrary length $z$, what is the amplification factor (power gain) of the signal beam?

   What happens if signal and idler beams both have 0 power at the input?

c) In practice, we consider that the medium is a negative uniaxial bulk crystal. Explain how to achieve **type I** (co-polarized $s$ and $i$ fields) and **type II** (cross-polarized $s$ and $i$ fields) phase matching. You may draw the two situations in a diagram showing the dispersion of the refractive index and mentioning the polarization of each beam. You may consider that $\omega_i$ is close to $\omega_s$ to simplify the drawing (near degenerate DFG).

d) We now account for **non-zero phase mismatch**, i.e., $\Delta k \neq 0$, but keep the undepleted pump approximation. Perform the change of variable $\alpha_{s,i}(z) = a_{s,i}(z)\, \exp\left(i\frac{\Delta k}{2}z\right)$ and show that the coupled equations can be separated in two decoupled second-order equations

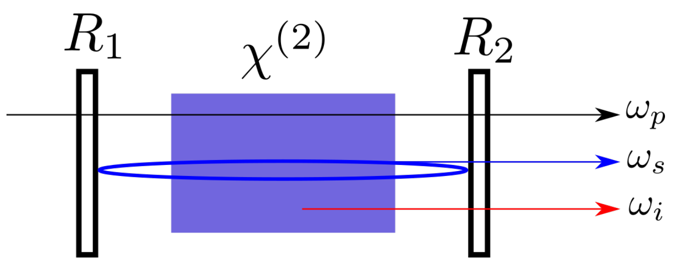
$$\frac{\partial^2 \alpha_{s,i}}{\partial z^2} - \gamma^2 \alpha_{s,i} = 0$$

   with $\gamma^2 = |g|^2 - (\Delta k)^2/4$.

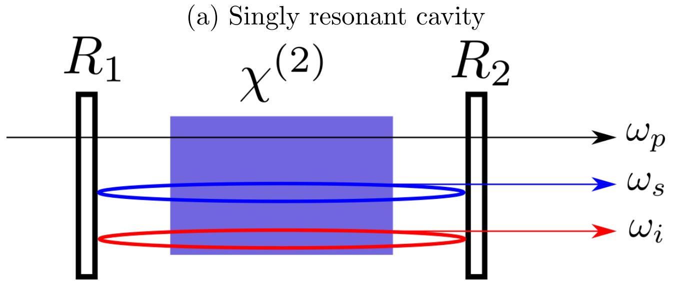
e) For a given phase mismatch, discuss the behavior of the solutions depending on the pump power.

## 1.3 Optical Parametric oscillator (OPO)

Based on the single-pass amplifier described above, it is possible to build an **optical parametric oscillator** by embedding the nonlinear crystal inside a cavity. We consider here a Fabry-Perot cavity, whose mirrors have negligible reflectivity for the pump beam.



(a) Singly resonant cavity



(b) Doubly resonant cavity

a) Is the light amplified when it travels in both directions through the crystal?

b) Consider a Fabry-Pérot cavity of length $L$ (equal to the crystal length) that is resonant at the signal frequency $\omega_s$ only (Fig. a). We assume perfect phase matching and a fractional photon round-trip loss $\eta_s$ at $\omega_s$ (i.e., for a circulating photon flux $\phi_s$ the rate of photon loss is $\eta_s \phi_s$).

Write the expression of the gain factor per round-trip. Deduce the 'threshold condition', which corresponds to a net amplification per round-trip.

c) Simplify the threshold condition for small ($\ll 1$) gain and loss per round-trip. Interpret this result.

d) Now, consider a cavity that is doubly resonant (at $\omega_s$ and $\omega_i$). In the limit of small loss and gain per round-trip, derive an expression for the variations of the amplitudes $\Delta a_{s,i}$ over one round-trip.

e) We define the threshold condition as $\Delta a_s = \Delta a_i = 0$. Justify this condition from the Manley-Rowe relation.

f) Nontrivial solutions satisfying $\Delta a_s = \Delta a_i = 0$ only exist when a determinant is zero. Show that it implies a relation between $(gL)^2$ and $\eta_{s,i}$.

g) Compare this threshold condition to the singly-resonant OPO threshold. Briefly explain some advantages and disadvantages of either system.