Week 05, problem 2 : Energy transfer

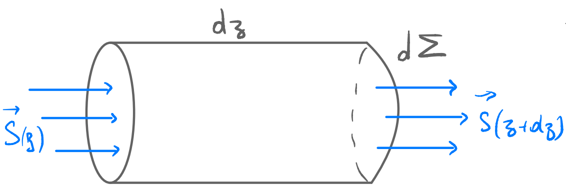
$$\text{(M2)} \quad \vec{\nabla} \times \vec{E} = -\mu_0 \dot{\vec{H}}$$
$$\text{(M3)} \quad \vec{\nabla} \times \vec{H} = \varepsilon_0 \dot{\vec{E}} + \dot{\vec{P}}$$

$$\hookrightarrow \; -\vec{\nabla}\cdot\vec{S} = \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$
$$= +\mu_0 \vec{H}\cdot\dot{\vec{H}} + \varepsilon_0 \vec{E}\cdot\dot{\vec{E}} + \vec{E}\cdot\dot{\vec{P}}$$

energy density $J.m^{-3}$

$$= \frac{\partial u}{\partial t} + \vec{E}\cdot\frac{\partial \vec{P}}{\partial t} \quad \text{where} \quad u(t) = \frac{1}{2}\left(\varepsilon_0 \vec{E}^{\,2} + \mu_0 \vec{H}^{\,2}\right)$$

We now consider the stationary regime where $\langle u(\vec{r},t)\rangle_t = u(\vec{r}) \rightarrow \frac{\partial u}{\partial t} = 0$



The magnitude of Poynting's vector is the wave intensity $I(z) = \|\vec{S}(z)\|$ $[W.m^{-2}]$

$$-\int \langle \vec{S}\rangle_t \cdot d\vec{n} = \int_V \left\langle \vec{E}\cdot\frac{\partial \vec{P}}{\partial t}\right\rangle_t dV$$

$$\Rightarrow \; -\left[I(z+dz) - I(z)\right] d\Sigma = \left\langle \vec{E}\cdot\frac{\partial \vec{P}}{\partial t}\right\rangle d\Sigma\, dz$$

Conclusion : in a lossless medium in the stationary regime :

$$\frac{dI}{dz} = -\left\langle \vec{E}\cdot\frac{\partial \vec{P}}{\partial t}\right\rangle \quad \text{where } \langle\cdot\cdot\rangle = \text{average over many optical cycles}$$

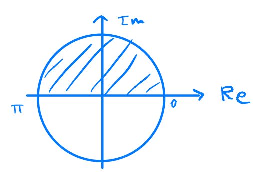
<u>In complex notation ?</u> $\quad E_i(t) = \frac{1}{2}\mathcal{E}_i e^{-i\omega t} + c.c.$ and $P_i(t) = \frac{1}{2}\mathcal{P}_i e^{-i\omega t} + c.c.$

$$\hookrightarrow \left\langle \vec{E}\cdot\frac{\partial \vec{P}}{\partial t}\right\rangle = \frac{1}{4}\left\langle -i\omega\, \mathcal{E}_i \mathcal{P}_i e^{-2i\omega t} + i\omega\, \mathcal{E}_i^* \mathcal{P}_i^* e^{2i\omega t}\right\rangle$$

$\mathcal{E} \in \mathbb{R}$ (ref phase)

$\mathcal{P} \in \mathbb{C}$

$$+ \frac{1}{4}\left\langle i\omega\, \mathcal{E}_i \mathcal{P}_i^* - i\omega\, \mathcal{E}_i^* \mathcal{P}_i\right\rangle = \frac{\omega}{2}\,\mathrm{Im}\left[\vec{\mathcal{E}}^*\cdot\vec{\mathcal{P}}\right]$$



<u>Case A</u> : <u>Linear medium</u> : $\mathcal{P}_i = \varepsilon_0 \chi_{ij}^{(1)} \mathcal{E}_j$

$$\frac{dI}{dz} = -\frac{\omega}{2}\,\mathrm{Im}\left[\mathcal{E}_i^*\,\varepsilon_0\,\chi_{ij}^{(1)}\,\mathcal{E}_j\right]$$ in the principal axes $\chi_{ij} = \delta_{ij}\chi_i$

$$= -\frac{\omega}{2}\varepsilon_0\,\mathrm{Im}\left[\chi_{ii}^{(1)}\,\mathcal{E}_i^*\,\mathcal{E}_i\right] = -\omega\sum_i \mathrm{Im}\left[\chi_{ii}^{(1)}\right]\frac{\varepsilon_0}{2}\left|\mathcal{E}_i\right|^2$$

We assume $\mathrm{Im}\left[\chi_{ii}^{(1)}\right] = \mathrm{Im}\left[\chi^{(1)}\right] \quad \forall\, i = x, y, z$

Then $\frac{dI}{dz} = -\alpha\, I(z)$ with $I(z) = \frac{1}{2}\varepsilon_0 n c\,|\vec{\mathcal{E}}|^2$ and $\boxed{\alpha = \frac{1}{n}\frac{\omega}{c}\,\mathrm{Im}\,\chi^{(1)}}$ absorption coeff.

Case B : $\chi^{(3)}$ medium $\quad \mathcal{P}_\omega^{(3)} = \frac{3}{4}\varepsilon_0\,\chi^{(3)}(-\omega_\sigma;\omega,-\omega,\omega)\,|\mathcal{E}_\omega|^2\mathcal{E}_\omega$ (opt. Kerr eff)

$$\frac{dI}{dz} = -\frac{3\,\omega}{\varepsilon_0 n^2 c^2}\,\mathrm{Im}\,\chi^{(3)}\,\frac{\varepsilon_0^2}{4}n^2c^2\,|\mathcal{E}|^4 = -\alpha_2 I^2$$

with $\boxed{\alpha_2 = \frac{3\omega}{\varepsilon_0 n^2 c^2}\,\mathrm{Im}\,\chi^{(3)}}$ $\rightarrow$ 2-photon absorption coefficient.

Case C : $\chi^{(2)}$ medium $\quad \mathcal{P}^{(2)}(\omega_\sigma) = \varepsilon_0\,\chi^{(2)}(-\omega_\sigma;\omega_1,\omega_2)\,\mathcal{E}(\omega_1)\mathcal{E}(\omega_2)$

$$\hookrightarrow \frac{dI(\omega_\sigma)}{dz} = -\frac{\omega}{2}\varepsilon_0\,\mathrm{Im}\left[\chi^{(2)}\,\mathcal{E}(\omega_1)\mathcal{E}(\omega_2)\mathcal{E}^*(\omega_1+\omega_2)\right] \neq f\left(I(\omega_\sigma)\right)$$

$\rightarrow$ We will see that fields exchange energy without loosing it to matter.