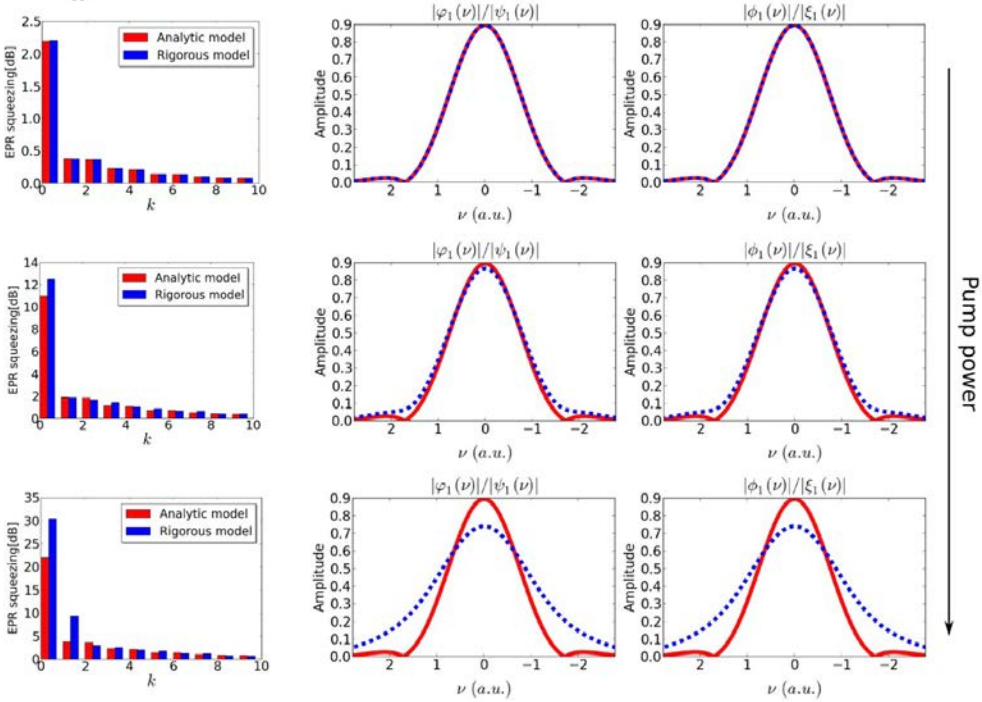
Remarks on Time-ordering

$$\hat{U}_{\mathrm{FC}} = \mathcal{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\, \hat{H}_{\mathrm{FC}}(t)\right]$$

Theory of quantum frequency conversion and type-II parametric down-conversion in the high-gain regime
Andreas Christ *et al* (2013) ***New J. Phys.*** 15 053038 https://iopscience.iop.org/article/10.1088/1367-2630/15/5/053038

$-10\log_{10}(e^{-2r_k})$



$\langle n \rangle = \sum_k \sinh^2(r_k)$

$\langle n \rangle = 0.07/0.07$

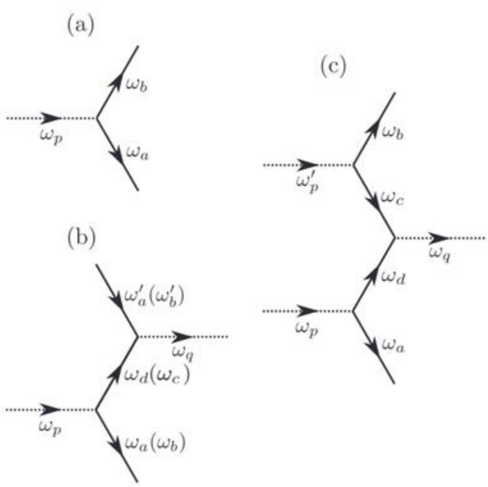
$\langle n \rangle = 2.80/4.08$

$\langle n \rangle = 39.39/279.87$

Remarks on Time-ordering

Effects of time ordering in quantum nonlinear optics
N. Quesada and J. E. Sipe Phys. Rev. A 90, 063840 (2014)

Diagrams representing the terms of increasing order in the Magnus expansion



- It is necessary to go to third order in the expansion to see a correction to the SPDC and SFWM states
- Corrections due to time ordering vanish exactly if the phase-matching function is sufficiently broad

Summary of Week 11

Under the following approximations

* Neglect losses (already needed for canonical quantize)
* Low gain regime (< 1 photon pair emitted per pulse on average)
* Quasi-monochromatic approx. (> 100 fs pulse duration)
  ↳ neglect dispersion withing each mode
* Strong pump → classical, undepleted

↳ weak nonlinearity

The effective interaction Hamiltonian :

$$\mathcal{H}_I = i\hbar g \int d\omega_s \int d\omega_i \, f(\omega_s, \omega_i) \, \hat{a}_s^\dagger(\omega_s) \, \hat{a}_i^\dagger(\omega_i) + h.c.$$

↳ TE_{00}, TM_{00} ...

with the gain $\boxed{|g|^2 \propto N_p |\chi_{eff}^{(2)}|^2 \propto \frac{N_p}{\mathcal{A}} |\chi_{eff}^{(2)}|^2}$

and the joint spectral amplitude (JSA) $\boxed{f(\omega_s, \omega_i) = \tilde{\eta}(\omega_s, \omega_i) \times \alpha(\omega_s + \omega_i)}$

phase matching function   pump enveloppe

6) Biphoton wave-function

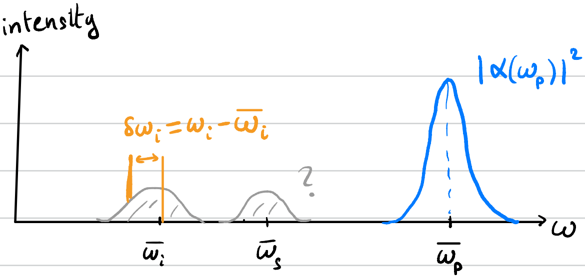
Initially : $|\Psi(t=-\infty)\rangle_{s,i} = |vac\rangle_{s,i}$

To first order in g : $\hat{U} = \exp\left(-\frac{i}{\hbar} \mathcal{H}_I\right) \simeq \mathbb{1} - \frac{i}{\hbar} \mathcal{H}_I$

Thus : $$|\Psi\rangle_{s,i} = \hat{U}|vac\rangle = |vac\rangle + g \underbrace{\boxed{\int d\omega_s \int d\omega_i \, f(\omega_s, \omega_i) \, |\omega_s, \omega_i\rangle}}_{\text{Biphoton wave function}}$$

Here $|\omega_s, \omega_i\rangle = |\omega_s\rangle_s \otimes |\omega_i\rangle_i$ with $|\omega_s\rangle_s \equiv \hat{a}_s^\dagger(\omega_s) |vac\rangle$

II) Joint Spectral Amplitude



We assume that $\boxed{\bar{k}_p - \bar{k}_s - \bar{k}_i = 0}$

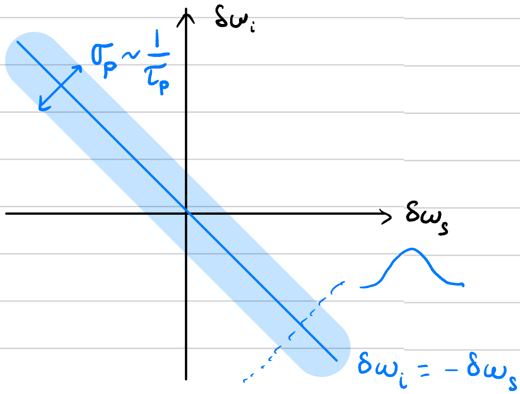
$$\bar{k}_p = k_p(\bar{\omega}_p)$$

Def.: Joint spectral Intensity $\quad |f(\omega_s,\omega_i)|^2 = |\alpha(\omega_s+\omega_i)|^2 \cdot |\tilde{\eta}(\omega_s,\omega_i)|^2$

Normalized : $\int d\omega_s \int d\omega_i \, |f(\omega_s,\omega_i)|^2$

1) Pump spectrum

Gaussian pulse : $\alpha(\omega_s+\omega_i) = \exp \dfrac{(\omega_s+\omega_i-\bar{\omega}_p)^2}{\sigma_p^2} = \exp \dfrac{(\delta\omega_s+\delta\omega_i)^2}{\sigma_p^2}$



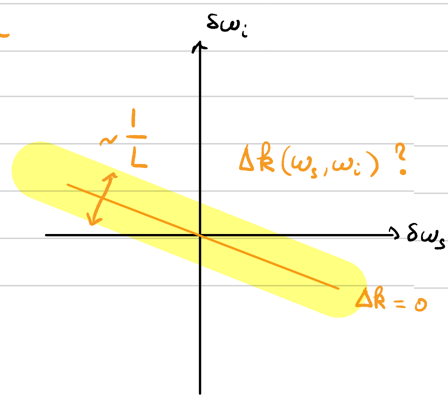
Shorter pulse $\leftrightarrow$ larger σ_p

2) Phase matching

Typically : $\tilde{\eta}(\Delta k) \propto \mathrm{sinc}\left(\frac{L}{2}\Delta k\right)$ — L: waveguide length

To first order around $\bar{\omega}_s, \bar{\omega}_i$:

$$k_s(\omega_s) = \bar{k}_s + \delta\omega_s \left.\frac{\partial k_s}{\partial \omega_s}\right|_{\bar{\omega}_s} = \bar{k}_s + k_s' \, \delta\omega_s$$



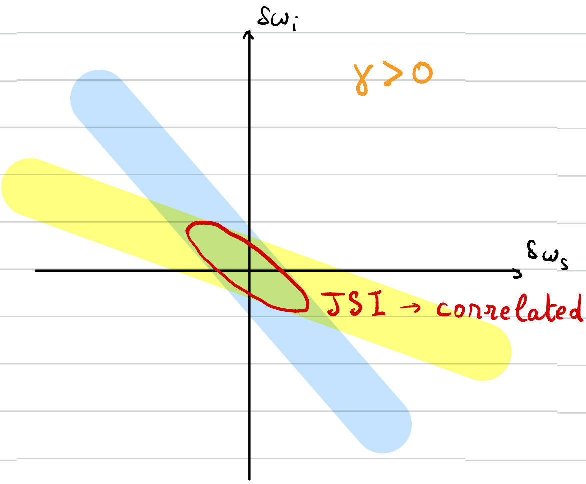
where $k_s' = \dfrac{1}{v_g^{(s)}}$ with $v_g^{(s)} = \dfrac{\partial \omega_s}{\partial k_s}\Big|_{\bar{\omega}_s}$ is the group velocity

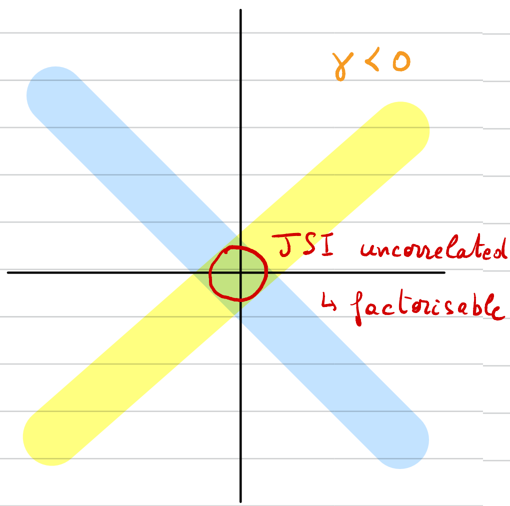
Then : $\Delta k = k_p' \delta\omega_p - k_s' \delta\omega_s - k_i' \delta\omega_i$ but $\delta\omega_p = \delta\omega_s + \delta\omega_i$

$$= \delta\omega_s (k_p' - k_s') + \delta\omega_i (k_p' - k_i')$$

Thus $\Delta k = 0 \iff \boxed{\delta\omega_i = -\gamma\, \delta\omega_s}$ where $\boxed{\gamma = \dfrac{k_p' - k_s'}{k_p' - k_i'}}$

- If $k_p' > k_s', k_i'$ then $\gamma > 0$
- If $k_s' > k_p' > k_i'$ / $k_i' > k_p' > k_s'$ then $\gamma < 0$



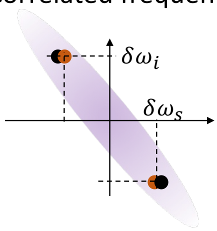


$f(\omega_s, \omega_i) = f^{(s)}(\omega_s) \times f^{(i)}(\omega_i)$

Joint Spectral Amplitude and non-separability

Toy model: comparing joint spectral intensities and the resulting heralded single photon states

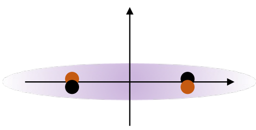
Correlated frequencies $\quad |\psi\rangle_{s,i} = |\delta\omega_s\rangle \otimes |-\delta\omega_i\rangle + |-\delta\omega_s\rangle \otimes |\delta\omega_i\rangle, \qquad \hat{\rho}_{s,i} = |\psi\rangle\langle\psi|$



$$\mathrm{Tr}_i\{\hat{\rho}_{s,i}\} = \langle\delta\omega_i|\hat{\rho}_{s,i}|\delta\omega_i\rangle + \langle-\delta\omega_i|\hat{\rho}_{s,i}|-\delta\omega_i\rangle$$
$$= |-\delta\omega_s\rangle\langle-\delta\omega_s| + |\delta\omega_s\rangle\langle\delta\omega_s|$$

Marginal state is mixed

Uncorrelated frequencies $\quad |\psi\rangle_{s,i} = |\delta\omega_s\rangle \otimes |0\rangle + |-\delta\omega_s\rangle \otimes |0\rangle$



$$= \underbrace{(|\delta\omega_s\rangle + |-\delta\omega_s\rangle)}_{\text{Marginal state is pure}} \otimes |0\rangle$$

In General, considering a pure joint state of two modes (or particles):
- If it is entangled (non-separable), the marginal states of each mode are mixed
- If it is separable, the marginal states are pure

Pioneering papers:

- *Continuous Frequency Entanglement: Effective Finite Hilbert Space and Entropy Control*

C. K. Law, I. A. Walmsley, and J. H. Eberly ***Phys. Rev. Lett.*** 84, 5304 (2000) https://doi.org/10.1103/PhysRevLett.84.5304

- *Eliminating frequency and space-time correlations in multiphoton states*

W. P. Grice, A. B. U'Ren, and I. A. Walmsley ***Phys. Rev. A*** 64, 063815 (2001) https://doi.org/10.1103/PhysRevA.64.063815

Joint spectral amplitude: tuning it

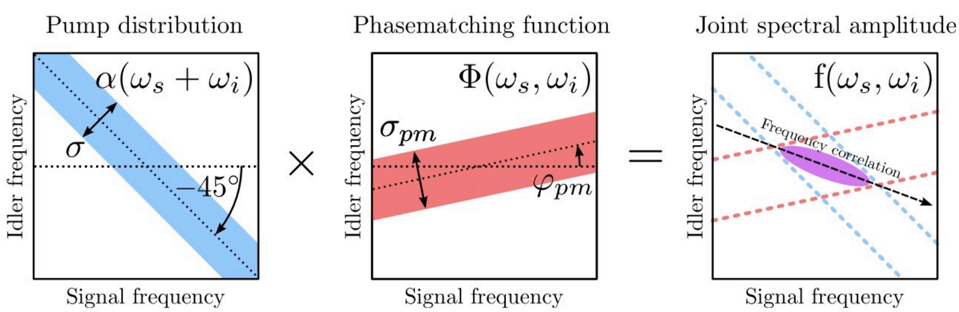


Figure 3.2: SPDC process with positive phasematching slope.

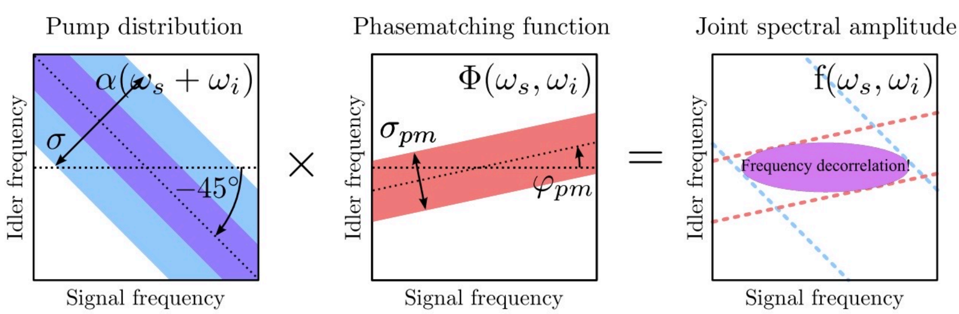


Figure 3.3: SPDC process with positive phasematching slope *and* matched pump width.

Andreas Eckstein PhD thesis (2012) *Kontrolle über Quantenlichtpulse durch nichtlineare Interaktion inWellenleitern*

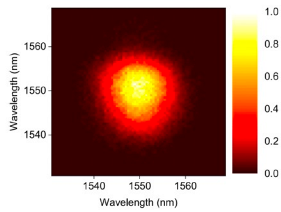
Example

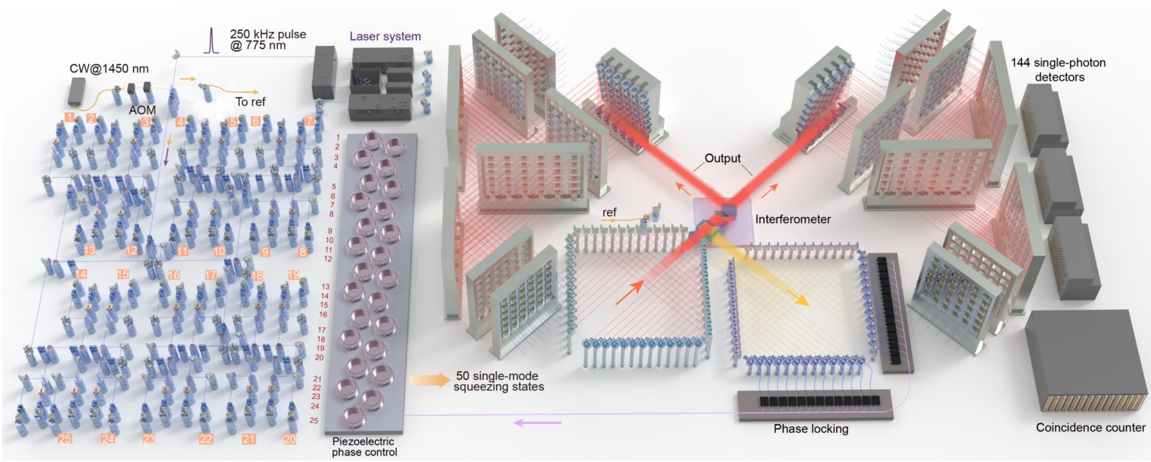
PHYSICAL REVIEW LETTERS **127,** 180502 (2021)

Editors' Suggestion | Featured in Physics

Phase-Programmable Gaussian Boson Sampling Using Stimulated Squeezed Light

This photonic quantum computer, Jiuzhang 2.0, yields a Hilbert space dimension up to $\sim 10^{43}$, and a sampling rate $\sim 10^{24}$ faster than using brute-force simulation on classical supercomputers.





Example

PHYSICAL REVIEW LETTERS **127,** 180502 (2021)

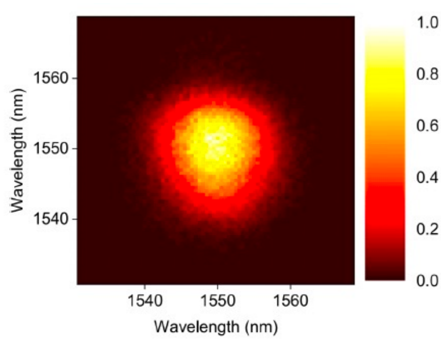
Editors' Suggestion | Featured in Physics

Phase-Programmable Gaussian Boson Sampling Using Stimulated Squeezed Light

"We customize the poling scheme of PPKTP crystals to eliminate the unwanted frequency correlation."

"We measured a spectral purity of 0.98 without using any narrowband filters, which indicates that almost all photons are in the Schmidt decomposition major mode."

$$2k_p'(\omega_{H0}+\omega_{V0}) = k_H'(\omega_{H0}) + k_V'(\omega_{V0})$$



For many quantum information/communication applications, we want **photons produced by several nonlinear crystals to interfere with each other;** i.e. they must be **indistinguishable** in all degrees of freedom.

It only occurs if the joint biphoton wavefunction at the output of each source is separable, meaning it contains no frequency correlations.

III) Full wavefunction after SPDC

$$|vac\rangle_{s,i} \longrightarrow \boxed{\hat{U}} \longrightarrow |\Psi\rangle_{s,i} = \hat{U}\,|vac\rangle \quad \text{where} \quad \hat{U} = \exp\left(-\frac{i}{\hbar}\mathcal{H}_I\right)$$

$\Uparrow$

$g \,;\, f(\omega_s, \omega_i)$

1) Schmidt decomposition (~ singular value decomposition in algebra)

We can always write $\boxed{f(\omega_s,\omega_i) = \sum_m \sqrt{\lambda_m}\, f_m^{(s)}(\omega_s) \times f_m^{(i)}(\omega_i)}$

with $\lambda_m \in \mathbb{R}^+$, $\sum_m \lambda_m = 1$, $\lambda_1 > \lambda_2 > \dots$ Temporal modes

and $\int d\omega_s\, f_m^{(s)*}(\omega_s)\, f_\ell^{(s)}(\omega_s) = \delta_{m\ell}$ same for $f_m^{(i)}(\omega_i)$

2) Schmidt number K

with $\boxed{\frac{1}{K} = \sum_m \lambda_m^2}$ $\leftrightarrow$ purity of the heralded single photon

Ex: For M equally populated pairs of modes $\lambda_m = \frac{1}{M}$ for $1 \leq m \leq M$

0 for $m > M$

Then $\frac{1}{K} = \sum_{m=1}^{M} \left(\frac{1}{M}\right)^2 = \frac{1}{M}$ so $\boxed{K = M}$

↳ in general, $\boxed{\text{K counts the number of excited mode pairs}}$

↳ For a given JSA, K is unique.

3) Evolution operator

$$\hat{U} = \exp\left(-\frac{i}{\hbar}\sum_m \hat{\mathcal{H}}_m\right) \quad \text{with} \quad \hat{\mathcal{H}}_m = i\hbar g\sqrt{\lambda_m}\, \underbrace{\int d\omega\, f_m^{(s)}(\omega)\,\hat{a}_s^\dagger(\omega)}_{\hat{A}_m^\dagger} \times \underbrace{\int d\omega\, f_m^{(i)}(\omega)\,\hat{a}_i^\dagger(\omega)}_{\hat{B}_m^\dagger} + h.c.$$

Creation operators for the Temporal modes

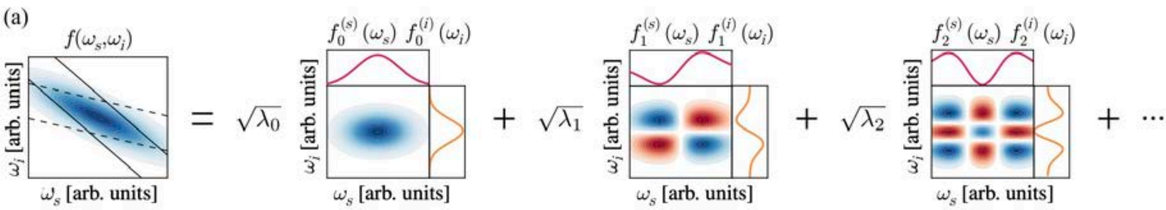
Schmidt decomposition

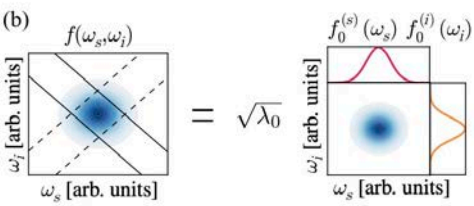
The Schmidt decomposition consists in writing the joint spectral amplitude (JSA) as a sum of factorised JSA with decreasing weights:

$$f(\omega_s,\omega_i) = \sum_m \sqrt{\lambda_m}\; f_m^{(s)}(\omega_s)\, f_m^{(i)}(\omega_i) \qquad \lambda_m > 0, \qquad \sum_m \lambda_m = 1$$

The Schmidt number K is defined as: $\frac{1}{K} = \sum_m {\lambda_m}^2$ = purity
(K measures the number of populated modes)

Graphical view:



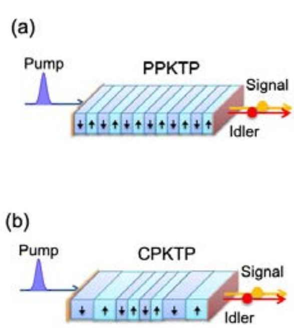


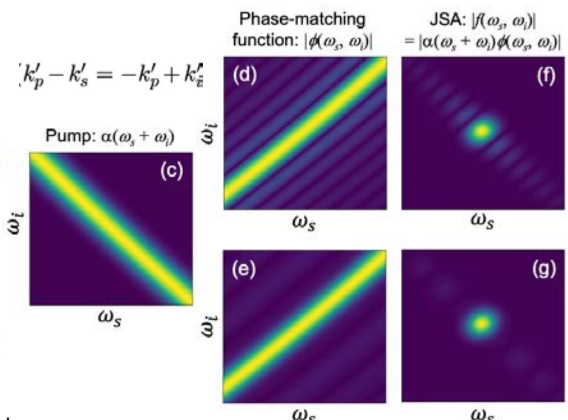
Photon Temporal Modes: A Complete Framework for Quantum Information Science; ***PRX*** 2015
http://dx.doi.org/10.1103/PhysRevX.5.041017

cf. also
Andreas Christ *et al* (2011) ***New J. Phys.*** 13 033027

Entangled quantum systems and the Schmidt decomposition
Artur Ekert and Peter L. Knight (1995) ***Am. J. of Phys.*** **63**, 415
https://doi.org/10.1119/1.17904

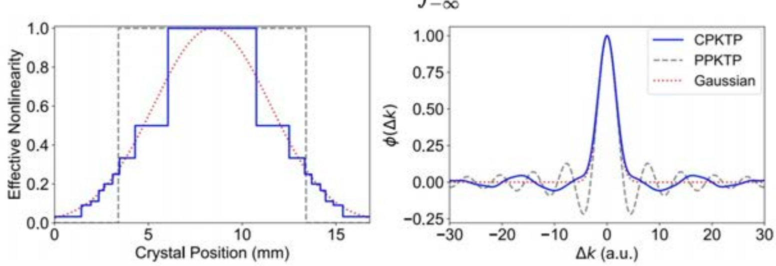
Using periodic poling to engineer the JSA





95% single-photon purity

Phase matching function is Fourier transform of the \chi^(2) profile: $\phi(\Delta k) = \int_{-\infty}^{\infty} dz \chi^{(2)}(z) e^{i\Delta k z}$



$$\chi^{(2)}(z) = \sum_{m=-\infty}^{\infty} \frac{2}{m\pi} \sin^2(\pi m D) \sin \frac{2\pi m}{\Lambda} z$$

$$\Delta k(\Omega_s, \Omega_i) = 2\pi m/\Lambda$$

https://arxiv.org/pdf/2111.10981.pdf
Generation of spectrally factorable photon pairs via multi-order quasi-phase-matched spontaneous parametric downconversion