SPDC in a waveguide

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Reminder: Linear Hamiltonian and mode expansion

After applying the canonical quantization procedure, an arbitrary real field can be expanded in a normal mode basis as

$$\hat{\boldsymbol{D}}(\boldsymbol{r},t) = \sum_\alpha \left(\hat{a}_\alpha(t)\boldsymbol{D}_\alpha(\boldsymbol{r}) + \hat{a}^\dagger_\alpha(t)\boldsymbol{D}^*_\alpha(\boldsymbol{r})\right) \quad (1)$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \sum_\alpha \left(\hat{a}_\alpha(t)\boldsymbol{B}_\alpha(\boldsymbol{r}) + \hat{a}^\dagger_\alpha(t)\boldsymbol{B}^*_\alpha(\boldsymbol{r})\right) \quad (2)$$

with the equal-time commutation relations

$$[\hat{a}_\alpha(t), \hat{a}_\beta(t)] = [\hat{a}^\dagger_\alpha(t), \hat{a}^\dagger_\beta(t)] = 0 \quad \text{and} \quad [\hat{a}_\alpha(t), \hat{a}^\dagger_\beta(t)] = \delta_{\alpha\beta}$$

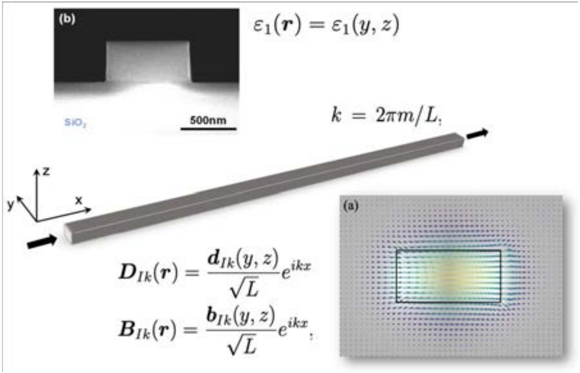
The linear medium Hamiltonian can then be expressed compactly as

$$H^{\mathrm{L}}(t) = \sum_\alpha \frac{\hbar\omega_\alpha}{2}(\hat{a}^\dagger_\alpha(t)\hat{a}_\alpha(t) + \hat{a}_\alpha(t)\hat{a}^\dagger_\alpha(t)) \longrightarrow \sum_\alpha \hbar\omega_\alpha \hat{a}^\dagger_\alpha(t)\hat{a}_\alpha(t)$$

where we used the commutation rule and removed the zero-point energy from the last expression.

Example: waveguide modes

A straight waveguide along x is characterized by the permittivity $\varepsilon(\boldsymbol{r}) = \varepsilon(y, z)$:



For a finite length L and periodic boundary conditions along x, the modes can be decomposed as

x, y, z

$$\boldsymbol{D}_{Tk}(\boldsymbol{r}) = \boldsymbol{d}_{Tk}(y,z)\frac{e^{ikx}}{\sqrt{L}} \quad \text{and} \quad \boldsymbol{B}_{Tk}() = \boldsymbol{b}_{Tk}(y,z)\frac{e^{ikx}}{\sqrt{L}}$$

mode index

wavevector

Example: waveguide modes

which is a product of a "one-dimensional plane wave" with wave-vector k along x and a two-dimensional mode profile $\boldsymbol{d}_{Tk}(y,z), \boldsymbol{b}_{Tk}(y,z)$ in the transverse plane, with mode index labeled by T. Note that the mode profile explicitly depends on k (geometric dispersion).

Any general solution of Maxwell's equation can then be expanded as

$$\hat{\boldsymbol{D}}(\boldsymbol{r},t) = \sum_{T,k} \sqrt{\frac{\hbar\omega_{Tk}}{2L}}\, \hat{a}_{Tk}(t)\boldsymbol{d}_{Tk}(y,z)e^{ikx} + \mathrm{H.c.} \tag{3}$$

and similarly for $\hat{\boldsymbol{B}}$. Next, we take the continuum limit when $L \longrightarrow \infty$ by substituting

$$\frac{2\pi}{L}\sum_{k} \longleftrightarrow \int \mathrm{d}k \quad \text{and} \quad \hat{a}_{Tk} \longleftrightarrow \sqrt{\frac{2\pi}{L}}\hat{a}_T(k)$$

where the last substitution is to conserve the proper commutation rules

$$[\hat{a}_T(k,t), \hat{a}^\dagger_{T'}(k',t')] = \delta_{TT'}\,\delta(k-k')\,\delta(t-t')$$

Example: waveguide modes

The final form of the field operators expanded on the waveguide modes is

$$\begin{aligned}
\hat{\boldsymbol{D}}(\boldsymbol{r}, t) &= \sum_T \int \mathrm{d}k \sqrt{\frac{\hbar\omega_{Tk}}{4\pi}}\, \hat{a}_T(k,t)\, \boldsymbol{d}_T(y,z,k) e^{ikx} + \mathrm{H.c.} &\quad (4)\\
\hat{\boldsymbol{B}}(\boldsymbol{r}, t) &= \sum_T \int \mathrm{d}k \sqrt{\frac{\hbar\omega_{Tk}}{4\pi}}\, \hat{a}_T(k,t)\, \boldsymbol{b}_T(y,z,k) e^{ikx} + \mathrm{H.c.} &\quad (5)
\end{aligned}$$

and the linear Hamiltonian is

$$\hat{H}^{\mathrm{L}}(t) = \sum_T \int \hbar\omega_{Tk}\, \hat{a}_T^\dagger(k,t) \hat{a}_T(k,t)\, \mathrm{d}k$$

Nonlinear interaction Hamiltonian

Neglecting dispersion in the nonlinear response, the displacement field writes

$$D_i(\boldsymbol{r},t) = \epsilon_0 \varepsilon_r(\boldsymbol{r}) E_i(\boldsymbol{r},t) + P_i^{(2)}(\boldsymbol{r},t) = \epsilon_0 \varepsilon_r(\boldsymbol{r}) E_i(\boldsymbol{r},t) + \epsilon_0 \chi_{ijk}^{(2)}(\boldsymbol{r}) E_j(\boldsymbol{r},t) E_k(\boldsymbol{r},t)$$

Since we want to express the Hamiltonian in terms of D, this expression is inverted:

$$E_i(\boldsymbol{r},t) = \frac{D_j(\boldsymbol{r},t)}{\epsilon_0 \varepsilon_r(\boldsymbol{r})} - \frac{\Gamma_{ijk}^{(2)}(\boldsymbol{r})}{\epsilon_0} D_j(\boldsymbol{r},t) D_k(\boldsymbol{r},t)$$

with

$$\Gamma_{ijk}^{(2)}(\boldsymbol{r}) = \frac{\chi_{ijk}^{(2)}(\boldsymbol{r})}{\epsilon_0 \varepsilon_r^3(\boldsymbol{r})}$$

from which the nonlinear interaction Hamiltonian can be obtained

$$\hat{H}_I(t) = -\frac{1}{3\epsilon_0} \int \Gamma_{ijk}^{(2)} : \hat{D}_i(\boldsymbol{r},t) \hat{D}_j(\boldsymbol{r},t) \hat{D}_k(\boldsymbol{r},t) : \mathrm{d}^3\boldsymbol{r} \qquad (6)$$

where : ... : indicates normal ordering of the creation and annihilation operators.

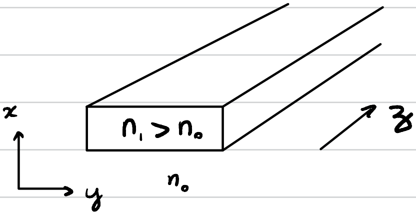
$a_1^\dagger a_2^\dagger a_3 + a_3^\dagger a_2 a_1$

Goal : initial state → * pump = coherent state (→ classical)

* signal & idler = vacuum

→ final state ?

1) Waveguide :



Modes are labeled by "μ"

Each component of the $\vec{D}$ field :

$$\hat{D}_j(\vec{r},t) = \sum_{\mu} \int dk \sqrt{\frac{\hbar\,\omega_{\mu,k}}{4\pi}}\; \overset{\nearrow e^{-i\omega_{\mu,k} t}}{\hat{a}_{\mu,k}(t)}\; \underbrace{\vec{d}_{\mu,k}(x,y)\cdot\vec{e}_j}_{\text{mode profile}}\; \underset{\text{propag. along } z}{e^{ikz}} + h.c. \quad \ni \hat{a}^{\dagger}\vec{d}^{*} e^{-ikz}$$

$d_{\mu k, j}$

2) Nonlinear interaction Hamiltonian

$$\hat{H}_I^{(2)}(t) = -\frac{1}{3\varepsilon_0}\int_V \underbrace{T^{(2)}_{ijl}}_{\propto\,\chi^{(2)}}(\vec{r})\;\hat{D}_i(\vec{r},t)\,\hat{D}_j(\vec{r},t)\,\hat{D}_l(\vec{r},t)\; d^3\vec{r}$$

$$= -\frac{1}{3\varepsilon_0}\int dz \int dx \int dy\; T^{(2)}_{ijl}(\vec{r}) \sum_{\mu_1\mu_2\mu_3} \iiint dk_1\,dk_2\,dk_3 \sqrt{\frac{\hbar^3\,\omega_{\mu_1 k_1}\,\omega_{\mu_2 k_2}\,\omega_{\mu_3 k_3}}{(4\pi)^3}} \times$$

$$\Big\{ \hat{a}^{(+)}_{\mu_1 k_1}\,\hat{a}^{(+)}_{\mu_2 k_2}\,\hat{a}^{(+)}_{\mu_3 k_3}\; e^{iz(\pm k_1 \pm k_2 \pm k_3)}\; e^{-it(\pm\omega_{\mu_1 k_1} \pm \omega_{\mu_2 k_2} \pm \omega_{\mu_3 k_3})}$$

$$d^{(*)}_{\mu_1 k_1; i}(x,y)\; d^{(*)}_{\mu_2 k_2; j}(x,y)\; d^{(*)}_{\mu_3 k_3; l}(x,y) \Big\} \rightarrow 8 \text{ terms}$$

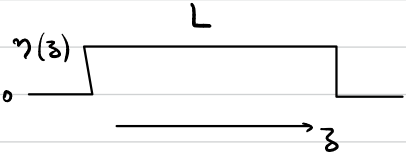
Spatial dependence of $T^{(2)}_{ijl}(\vec{r}) = T^{(2)}_{ijl}(x,y)\,\eta(z)$

* Integral over z → phase matching

$$\boxed{\int dz\; \eta(z)\, e^{iz\Delta k} = \tilde{\eta}(\Delta k)}$$ where $\Delta k = \pm k_1 \pm k_2 \pm k_3$

↳ phase matching function (Fourier Transform of η)

Ex :



$\rightarrow \tilde{\eta}(\Delta k) = \frac{L}{2}\,\mathrm{sinc}\left(\Delta k \frac{L}{2}\right)$

* Integral over x, y → nonlinear mode overlap

mode profiles

$$\gamma^{(2)}_{\substack{\mu_1 \mu_2 \mu_3 \\ k_1 k_2 k_3}} = \iint dx\, dy \sum_{ijl} T^{(2)}_{ijl}(x,y)\, d^{(*)}_{\mu_1 k_1, i}(x,y)\, d^{(*)}_{\mu_2 k_2, j}(x,y)\, d^{(*)}_{\mu_3 k_3, l}(x,y)$$

$\in (x\,y\,z)$

(8 versions of it)

$= \chi^{(2)}_{eff}$

$\propto \frac{1}{\sqrt{A}}$ where A is an effective nonlinear interaction area.

* Integration over time → time evolution of the wave function.

For an initial state $|\Psi_0\rangle$: $|\Psi(t)\rangle = \hat{U}(t)\,|\Psi_0\rangle$

where $\hat{U}(t) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t} \mathcal{H}^{(2)}_I(t')\,dt'\right)$ evolution operator

↓

time ordering operator

↓

no analytical solution

3) Low gain regime → we neglect time ordering

When $t \rightarrow +\infty$ the time integral gives $\int_{-\infty}^{+\infty} e^{-it'\Delta\omega}\,dt' = 2\pi\,\delta(\Delta\omega)$

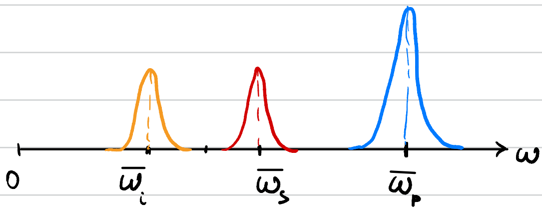
↳ $\hat{U}$ becomes $\hat{U} = \exp\left\{-\frac{i}{\hbar}\tilde{\mathcal{H}}_I\right\}$ where

energy conservation.

$$\boxed{\tilde{\mathcal{H}}_I = -\frac{2\pi}{3\varepsilon_0}\sum_{\mu's}\int dk's\; \gamma_{\substack{\mu's \\ k's}} \sqrt{\frac{\cdots}{\cdots}}\; \tilde{\eta}(\Delta k)\; \delta(\Delta\omega)\; \hat{a}^{(+)}_{\mu_1 k_1}\, \hat{a}^{(+)}_{\mu_2 k_2}\, \hat{a}^{(+)}_{\mu_3 k_3}}$$

4) Quasi-monochromatic approximation (for pulse durations > 100 fs)

We consider that phase matching ($\Delta k = 0$) is satisfied for a triplet of modes $\mu = p, s, i$ with center frequencies $\bar{\omega}_p$, $\bar{\omega}_s$, $\bar{\omega}_i$



Energy conservation selects 2 terms:

$$\boxed{\Delta\omega = \pm\left(\omega_p - \omega_s - \omega_i\right)} \quad \hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p$$

$\times$ 6 permutations of dummy indices.

Finally, we switch from $\int dk$ to $\int d\omega$

$$\omega = \frac{c}{n} k \rightarrow d\omega = \frac{c}{n} dk \quad \text{and} \quad \hat{a}(\omega) = \sqrt{\frac{n}{c}}\, \hat{a}(k) \quad \text{to conserve commutation rules}$$

$$\tilde{\mathcal{H}}_I = -\frac{4\pi}{\varepsilon_0} \sqrt{\frac{\hbar^3 \bar{k}_p \bar{k}_s \bar{k}_i}{(4\pi)^3}}\, \chi^{(2)}_{eff} \iiint d\omega_p\, d\omega_s\, d\omega_i\, \tilde{\eta}(\Delta k)\, \delta(\Delta\omega)\, \hat{a}_s^\dagger(\omega_s)\, \hat{a}_i^\dagger(\omega_i)\, \hat{a}_p(\omega_p) + h.c.$$

5) Parametric approximation : pump pulse = strong coherent state.

$$\hat{a}_p(\omega_p) \longrightarrow \sqrt{N_p}\, \alpha(\omega_p) = \text{complex number}$$

N_p: mean photon number per pulse

$\alpha(\omega_p)$: pulse enveloppe (~ Gaussian)

Since the integral $\int d\omega_p \ldots \delta(\Delta\omega)$ yields $\omega_p = \omega_i + \omega_s$, we eventually replace $\hat{a}_p(\omega_p)$ by $\sqrt{N_p}\, \alpha(\omega_i + \omega_s)$