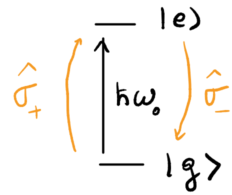
I) Quantum theory of detection

1) $\hat{H}_I(t) = -\frac{1}{\varepsilon_0}\,\hat{\vec{\mu}}\cdot\hat{\vec{D}}(\vec{r}_0, t)$ → two-level system located at $\vec{r}_0 = \vec{0}$

↳ dipole operator

$$\hat{\mu}_x = \mu_x(\hat{\sigma}_+ + \hat{\sigma}_-) \quad \text{where} \quad \hat{\sigma}_- = |g\rangle\langle e|$$

$$\hat{\sigma}_+ = |e\rangle\langle g|$$



Expanding over the modes:

$$\hat{H}_I(t) = \sum_\alpha \varepsilon_r\, \vec{\xi}_\alpha \cdot \vec{\mu}\,\left(\hat{a}_\alpha(t) + \hat{a}_\alpha^\dagger(t)\right)\cdot\left(\hat{\sigma}_+(t) + \hat{\sigma}_-(t)\right)$$

$$\vec{\xi}_\alpha = \sqrt{\frac{\hbar\omega_\alpha}{2\varepsilon_0\varepsilon_r V}}\,\vec{\epsilon}_\alpha$$

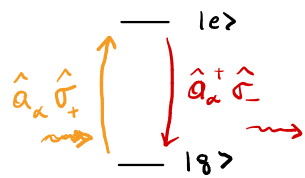
In the interaction picture, the operators evolves as:

$$\boxed{\hat{a}_\alpha(t) = \hat{a}_\alpha\, e^{-i\omega_\alpha t}} \quad \text{and} \quad \boxed{\hat{\sigma}_-(t) = \hat{\sigma}_-\, e^{-i\omega_0 t}}$$

* We assume that $\omega_\alpha \approx \omega_0$
* We time-average $\hat{H}_I(t)$ over many optical cycles $\left(T \gg \frac{2\pi}{\omega_0}\right)$

↳ Rotation wave approximation (RWA):

$$\hat{H}_I(t) \simeq \sum_\alpha \varepsilon_r\, \vec{\xi}_\alpha\cdot\vec{\mu}\,\Big(\underbrace{\hat{a}_\alpha\hat{\sigma}_+}_{\text{absorpt.}} + \underbrace{\hat{a}_\alpha^\dagger\hat{\sigma}_-}_{\text{emission}}\Big)$$



2) Transition probability : P_{det}^{α} (detecting a photon in mode α)

Initial state : $|I\rangle = |g\rangle_{atom} \otimes |i\rangle_{field}$

Final state : $|F\rangle = |e\rangle_{atom} \otimes |f\rangle_{field}$ $\rightarrow P_{I\to F} = \left|\langle F|\hat{H}_I|I\rangle\right|^2$

$$P_{I\to F} = \underbrace{\langle I|\hat{H}_I|F\rangle}\underbrace{\langle F|\hat{H}_I|I\rangle} \rightarrow \langle f|\hat{a}_\alpha|i\rangle$$

$$\langle i|\otimes\langle g|\,(\hat{a}_\alpha\hat{\sigma}_+ + \hat{a}_\alpha^\dagger\hat{\sigma}_-)\,|f\rangle\otimes|e\rangle = \langle i|\hat{a}_\alpha|f\rangle\cancel{\langle g|\hat{\sigma}_+|e\rangle}^{0} + \langle i|\hat{a}_\alpha^\dagger|f\rangle\underbrace{\langle g|\hat{\sigma}_-|e\rangle}_{=1}$$

$$= \langle i|\hat{a}_\alpha^\dagger|f\rangle$$

We have to sum over a complete set of final states for the field:

$$P_{det}^{\alpha} \propto \sum_f \langle i|\hat{a}_\alpha^\dagger|f\rangle\langle f|\hat{a}_\alpha|i\rangle = \boxed{\langle i|\hat{a}_\alpha^\dagger\,\hat{a}_\alpha|i\rangle}$$

$\hat{a}_\alpha^\dagger \rightarrow \hat{E}_\alpha^{(-)}$, $\hat{a}_\alpha \rightarrow \hat{E}_\alpha^{(+)}$

General expression :

$$\boxed{I(t) \propto \left\langle \hat{E}^{(-)}(t)\,\hat{E}^{(+)}(t)\right\rangle = \mathrm{Tr}\left\{\hat{\rho}_{field}\,\hat{E}^{(-)}\hat{E}^{(+)}\right\}}$$

time-averaged

Unnormalized first-order correlation (degree of 1st order coherence)

$$\boxed{G^{(1)}(\vec{r},\vec{r}\,',t,t') = \left\langle \hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r}\,',t')\right\rangle}$$

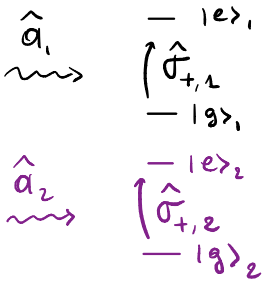
field autocorrelation

↳ If $\vec{r} = \vec{r}\,'$ and the state is stationary

$$\boxed{G^{(1)}(\tau) = \left\langle \hat{E}^{(-)}(\tau)\,\hat{E}^{(+)}(0)\right\rangle} \qquad G^{(1)}(-\tau) = G^{(1)}(\tau)^*$$

3) Two-photon detection probability

Consider two "detectors"



$$|I\rangle = |g\rangle_1 |g\rangle_2 \; |i\rangle_1 |i\rangle_2$$

P_{double} ?

$$|F\rangle = |e\rangle_1 |e\rangle_2 \; |f\rangle_1 |f\rangle_2$$

Need to look at second order processes in $\hat{H}_I^2$:

$$\langle F | \hat{H}_I^2 | I\rangle \propto \langle F | \hat{a}_1 \hat{a}_2 \hat{\sigma}_{+,1} \hat{\sigma}_{+,2} | I\rangle = \langle f|_{12} \hat{a}_1 \hat{a}_2 |i\rangle_{12}$$

$\hookrightarrow$ $P_{\text{double}} \propto \langle i | \hat{a}_1^+ \hat{a}_2^+ \hat{a}_1 \hat{a}_2 | i\rangle$

General expression :

Degree of second order coherence

$$\boxed{G^{(2)}(\vec{r},\vec{r}\,',t,t') = \left\langle \hat{E}^{(-)}(\vec{r},t) \hat{E}^{(-)}(\vec{r}\,',t') \hat{E}^{(+)}(\vec{r}\,',t') \hat{E}^{(+)}(\vec{r},t) \right\rangle}$$

$$\propto \left\langle \hat{a}_b^+ \hat{a}_a^+ \hat{a}_a \hat{a}_b \right\rangle$$

Often we use $\boxed{G^{(2)}(\tau) = \left\langle \hat{E}_b^{(-)}(0) \hat{E}_a^{(-)}(\tau) \hat{E}_a^{(+)}(\tau) \hat{E}_b^{(+)}(0) \right\rangle}$

$\hookrightarrow$ detection in mode "a" at t_0 & det. in mode "b" at $t_0 + \tau$

Normalized 2nd order correlation :

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{G^{(1)}(0)^2} = \frac{\langle \hat{a}^+(0) \hat{a}^+(\tau) \hat{a}(\tau) \hat{a}(0)\rangle}{\langle \hat{a}^+ \hat{a}\rangle^2}$$

$$g^{(2)}_{a,b}(\tau) = \frac{G^{(2)}_{a,b}(\tau)}{G^{(1)}_a(0)\, G^{(1)}_b(0)}$$

Intensity

$a = b$: auto-correlation

$a \neq b$: cross-correlation

When using single-photon counting module (SPCM)
= bucket detectors

$$\boxed{g^{(2)}(\tau) \simeq \frac{P_{coincidence}(\tau)}{P^2_{single}}}$$

→ source has low emission proba.
or
→ detection efficiency is low

4) Connection with the photon number operator

$$\langle \hat{n}(t) \rangle = \mathrm{Tr}\{\hat{\rho}\, \hat{a}^+(t)\, \hat{a}(t)\}$$

but $\langle \hat{a}^+ \hat{a}^+ \hat{a}\, \hat{a} \rangle = \langle \hat{a}^+ (\hat{a}\hat{a}^+ - 1)\, \hat{a} \rangle = \langle \hat{n}^2 - \hat{n} \rangle = \langle \hat{n}(\hat{n}-1) \rangle$

$\neq \langle \hat{n} \rangle \langle \hat{n} - 1 \rangle$

For the Fock state $|n=1\rangle$, $\hat{\rho} = |1\rangle\langle 1|$

$$\langle 1 | \hat{a}^+ \hat{a}^+ \hat{a}\, \hat{a} | 1 \rangle = 0 \quad \rightarrow \quad \boxed{g^{(2)}(0) = 0}$$

II) Signatures of non-classical states

1) For classical fields

i) $$G^{(1)}(\tau) = \langle E^*(t)\, E(t+\tau) \rangle_T$$

→ average over $T \gg \frac{2\pi}{\omega}$

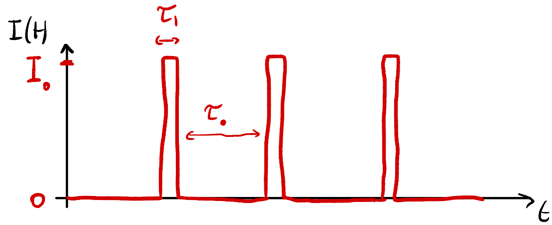
and $$g^{(1)}(\tau) = \frac{G^{(1)}(\tau)}{G^{(1)}(0)}$$

ii) $$G^{(2)}_{a,b}(\tau) = \langle E^*_a(t)\, E^*_b(t+\tau)\, E_b(t+\tau)\, E_a(t) \rangle_T = \langle I_a(t)\, I_b(t+\tau) \rangle$$

$$g^{(2)}_{a,b}(\tau) = \frac{G^{(2)}_{a,b}(\tau)}{G^{(1)}_a(0)\, G^{(1)}_b(0)}$$

in particular $g^{(2)}(0) = \dfrac{\langle I(t)^2 \rangle}{\langle I(t) \rangle^2}$

Ex :



$\frac{2\pi}{\omega} < \tau_0, \tau_1 < T_{avg}$

$$\langle I(t) \rangle = \frac{\tau_1}{\tau_0 + \tau_1} \cdot I_0 \quad \text{and} \quad \langle I(t)^2 \rangle = \frac{\tau_1}{\tau_0 + \tau_1} I_0^2 \Rightarrow g^{(2)}(0) = \frac{\tau_0 + \tau_1}{\tau_1} \xrightarrow[\tau_1 \to 0]{} +\infty$$

2) Cauchy - Schwarz inequalities (CSI)

Let $f(t)$ and $g(t)$ be square - integrable complex functions :

$$\boxed{\left| \int f(t) g^*(t) \, dt \right|^2 \leqslant \int |f(t)|^2 dt \int |g(t)|^2 dt} \quad \text{CSI}$$

a) Take $f(t) = I(t)$ and $g(t) = 1$ we get $\langle I(t) \rangle^2 \leqslant \langle I(t)^2 \rangle$

↳ for classical fields $\boxed{g^{(2)}(0) \geqslant 1}$

Three types of photon statistics :

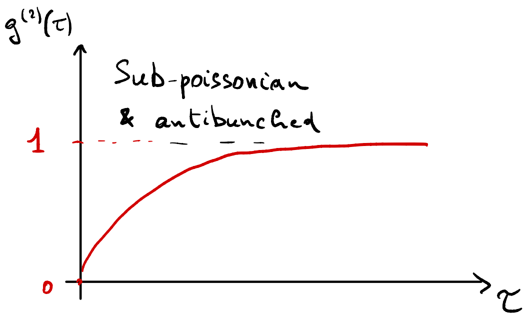
* $g^{(2)}(0) = 1 \rightarrow$ Poissonian statistics (ex : laser light)
* $g^{(2)}(0) > 1 \rightarrow$ Super poissonian (ex : Single mode thermal light $g^{(2)}(0) = 2$)
* $g^{(2)}(0) < 1 \rightarrow$ Sub - poissonian
  ↳ non - classical (ex : $n=1$ Fock state) $g^{(2)}(0) = 0$

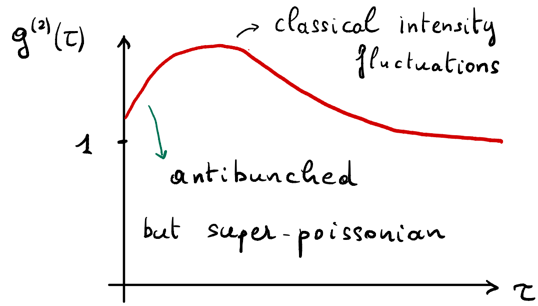
b) Take $f(t) = I(t)$ and $g(t) = I(t+\tau)$, from the CSI we get :

$$\frac{1}{T^2}\left(\int_T I(t)\, I(t+\tau)\, dt\right)^2 \leqslant \frac{1}{T}\int_T I(t)^2\, dt \cdot \frac{1}{T}\int_T I(t+\tau)^2\, dt$$
$$\leqslant \frac{1}{T^2}\left(\int_T I(t)^2\, dt\right)^2$$

thus $G^{(2)}(\tau) \leqslant G^{(2)}(0)$ and therefore $\boxed{g^{(2)}(0) \geqslant g^{(2)}(\tau)}$

$\boxed{\text{When } g^{(2)}(0) \leqslant g^{(2)}(\tau) \longrightarrow \text{anti-bunching}}$





c) CSI for two distinct optical modes a and b.

Cross correlation : $g^{(2)}_{a,b}(0) = \dfrac{\langle I_a(t)\, I_b(t)\rangle}{\langle I_a(t)\rangle\langle I_b(t)\rangle}$

from the CSI : $\boxed{g^{(2)}_{a,b}(0) \leqslant \sqrt{g^{(2)}_{a,a}(0)\, g^{(2)}_{b,b}(0)}}$

- We will see that light produced in SPDC can violate this CSI.
- If modes a and b are uncorrelated $\rightarrow g^{(2)}_{a,b}(0) = 1$.