# Canonical quantization of the electromagnetic field

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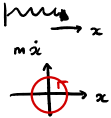
# Lagrangian mechanics

The Lagrangian[1] $L(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ is a function of the dynamical vectors $\boldsymbol{q} = \{q_j\}$ (where $j = 1 \ldots N$ runs over the system's $N$ degrees of freedom) and $\boldsymbol{v} = \dot{\boldsymbol{q}}$ in the $2N$-dimensional configuration space, constructed so as to recover the equations of motion when using the Euler-Lagrange equation:




$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_j} L\right) - \frac{\partial}{\partial q_j} L = 0 \tag{1}$$

The canonical momentum associated with each generalized coordinate $q_j(t)$ is

$$p_j(t) = \frac{\partial}{\partial \dot{q}_j} L$$

Example 1: For a massive but charge-less point particle, moving in the potential $U(\boldsymbol{r})$, the Lagrangian is

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - U(\boldsymbol{r})$$

(kinetic energy − potential energy) and the canonical momentum is

$$\boldsymbol{p} = m\dot{\boldsymbol{r}} = m\boldsymbol{v}$$

[1] In field theories, the Lagrangian is the volume integral of a Lagrangian density

# Lagrangian mechanics

Example 2: For a charged particle in an electromagnetic field

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad ; \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

where $\phi$ and $\boldsymbol{A}$ are the scalar and vector potentials, the Lagragian is

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 + q \dot{\boldsymbol{r}} \cdot \boldsymbol{A} - q\phi$$

and the canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + q\underline{\boldsymbol{A}} \neq m\boldsymbol{v}$$

The Euler-Lagrange equation yields the correct Lorentz-Coulomb force

$$m\ddot{\boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

# Hamiltonian mechanics

From the Lagrangian, we define a new function called the Hamiltonian

$$H(t, \boldsymbol{q}, \boldsymbol{p}) = \sum_{j=1}^{N} \dot{q}_j p_j - L(t, \boldsymbol{q}, \boldsymbol{p}) \tag{2}$$

One can show that the Euler-Lagrange equation (1) is equivalent to two sets of equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad ; \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{3}$$

Equation of Motion: Any function $f(t, \boldsymbol{q}, \boldsymbol{p})$ (e.g. an observable like the photon number, the field intensity, etc.) evolves in time according to

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} \left( + \frac{\partial f}{\partial t} \right)} \tag{4}$$

where we defined the Poisson brackets

$$\{f, H\} = \sum_j \left( \frac{\partial f}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \tag{5}$$

# Remark: connection with Lie algebra

A Lie algebra over $\mathbb{C}$ is a vectorial space $\mathcal{V}$ over $\mathbb{C}$ equipped with a Lie bracket, that is, a bilinear application $\{f, g\} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ that verifies

$$\begin{aligned} \{f, f\} &= 0 \quad \textit{Alternating property} \qquad (6) \\ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} &= 0 \quad \textit{Jacobi identity} \qquad (7) \end{aligned}$$

Bilinearity and the alternating property imply anti-commutativity

$$\{f, g\} = -\{g, f\}$$

Exercise: verify that the Poisson brackets defined in eq. 5 satisfy the properties of a Lie algebra over $\mathbb{C}$.

# Canonical quantization

Once a dynamical theory is expressed in the Hamiltonian formalism, canonical quantization simply consists in the substitution

$$\{\ ,\ \} \longrightarrow \frac{1}{i\hbar}[\ ,\ ] \quad \text{where} \quad [A, B] = AB - BA$$

In the case of electrodynamics in a dielectric medium, the generalized coordinate is the four-vector potential $(A_0 = \phi, \boldsymbol{A})$ from which the $\boldsymbol{E}$ and $\boldsymbol{B}$ field are derived as

$$\boldsymbol{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t} \quad ; \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

To recover Maxwell's equations, the appropriate Lagrangian density in a time-independent linear medium is[2]

$$L = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{8}$$

The canonical momentum is the four-vector $(\Pi_0, \mathbf{\Pi})$ with

$$\Pi_0 = \frac{\partial L}{\partial \dot{A}_0} = 0 \quad \text{and} \quad \Pi_j = \frac{\partial L}{\partial \dot{A}_j} = -D_j \text{ (displacement field)}$$

[2] *see, e.g., Quantization of electrodynamics in nonlinear dielectric media*, Mark Hillery, Leonard D. Mlodinow; Phys. Rev. A (1984)

# Linear Hamiltonian and mode expansion

In terms of $\boldsymbol{D}$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, the Hamiltonian in a linear medium is[3]

$$H^{\mathrm{L}}(t) = \frac{1}{2} \int_V \left( \frac{\boldsymbol{D}(\boldsymbol{r},t) \cdot \boldsymbol{D}(\boldsymbol{r},t)}{\epsilon_0 \varepsilon(\boldsymbol{r})} + \frac{\boldsymbol{B}(\boldsymbol{r},t) \cdot \boldsymbol{B}(\boldsymbol{r},t)}{\mu_0} \right) \mathrm{d}^3\boldsymbol{r} \tag{9}$$

where $V$ is the entire volume over which the field may exist. We want to expand the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ on a basis of stationary (= monochromatic) solutions of Maxwell's equations: the *normal modes* (for a loss-less system), satisfying

Coulomb gauge

$$\boxed{\boldsymbol{\nabla} \times \left[ \frac{\boldsymbol{\nabla} \times \boldsymbol{B}_\alpha(\boldsymbol{r})}{\varepsilon(\boldsymbol{r})} \right] = \frac{\omega_\alpha^2}{c^2} \boldsymbol{B}_\alpha(\boldsymbol{r})} \quad \text{and} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B}_\alpha(\boldsymbol{r}) = 0 \tag{10}$$

For each mode $\alpha$ the displacement field is uniquely defined as

$$\boldsymbol{D}_\alpha(\boldsymbol{r}) = \frac{i}{\mu_0\, \omega_\alpha} \boldsymbol{\nabla} \times \boldsymbol{B}_\alpha(\boldsymbol{r}) \quad \text{(Maxwell's eq.)} \tag{11}$$

The normal modes are orthogonal, and we also take them to be normalized as:

$$\int_V \frac{\boldsymbol{D}_\alpha^*(\boldsymbol{r}) \cdot \boldsymbol{D}_\beta(\boldsymbol{r})}{\epsilon_0 \varepsilon(\boldsymbol{r})} \mathrm{d}^3\boldsymbol{r} = \int_V \frac{\boldsymbol{B}_\alpha^*(\boldsymbol{r}) \cdot \boldsymbol{B}_\beta(\boldsymbol{r})}{\mu_0} \mathrm{d}^3\boldsymbol{r} = \frac{\hbar\omega_\alpha}{2} \delta_{\alpha\beta} \tag{12}$$

[3] The following notes follow: N. Quesada, L. G. Helt, M. Menotti, M. Liscidini, and J. E. Sipe, *Adv. Opt. Photon.* **14**, 291-403 (2022)

# Linear Hamiltonian and mode expansion

After applying the canonical quantization procedure, an arbitrary real field can be expanded in the normal mode basis as

classically: complex numbers

$$\hat{\boldsymbol{D}}(\boldsymbol{r},t) = \sum_{\alpha} \left( \hat{a}_{\alpha}(t)\boldsymbol{D}_{\alpha}(\boldsymbol{r}) + \hat{a}^{\dagger}_{\alpha}(t)\boldsymbol{D}^{*}_{\alpha}(\boldsymbol{r}) \right) \tag{13}$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \sum_{\alpha} \left( \hat{a}_{\alpha}(t)\boldsymbol{B}_{\alpha}(\boldsymbol{r}) + \hat{a}^{\dagger}_{\alpha}(t)\boldsymbol{B}^{*}_{\alpha}(\boldsymbol{r}) \right) \tag{14}$$

with the equal-time commutation relations

$$[\hat{a}_{\alpha}(t), \hat{a}_{\beta}(t)] = [\hat{a}^{\dagger}_{\alpha}(t), \hat{a}^{\dagger}_{\beta}(t)] = 0 \quad \text{and} \quad [\hat{a}_{\alpha}(t), \hat{a}^{\dagger}_{\beta}(t)] = \delta_{\alpha\beta}$$

The linear medium Hamiltonian can then be expressed compactly as

$$H^{\mathrm{L}}(t) = \sum_{\alpha} \frac{\hbar\omega_{\alpha}}{2} \left( \hat{a}^{\dagger}_{\alpha}(t)\hat{a}_{\alpha}(t) + \hat{a}_{\alpha}(t)\hat{a}^{\dagger}_{\alpha}(t) \right) \longrightarrow \sum_{\alpha} \hbar\omega_{\alpha} \left( \hat{a}^{\dagger}_{\alpha}(t)\hat{a}_{\alpha}(t) + \frac{1}{2} \right)$$

where we used the commutation rule and removed the zero-point energy from the last expression.

# Quantization in homogeneous medium (isotropic)

dispersionless.

$\varepsilon = \varepsilon_r \varepsilon_0$ (does not depend on $\vec{r}$, $\omega$)

Cube of length $L$, volume $V = L^3$ and periodic boundary conditions.

$$\vec{k}_\alpha = \left(\alpha_x \frac{2\pi}{L}, \alpha_y \frac{2\pi}{L}, \alpha_z \frac{2\pi}{L}\right) \qquad (\text{except } \alpha = 0,0,0)$$

↳ positive or negative integers

$$\text{We try with } \quad \vec{B}_\alpha(\vec{r}) = C_\alpha e^{i\vec{k}_\alpha\cdot\vec{r}} \vec{\epsilon}_\alpha \quad \text{with } \alpha = (\alpha_x, \alpha_y, \alpha_z, \underset{\substack{\uparrow \\ \in\{1,2\} \text{ polariz.}}}{\alpha_\epsilon})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}_\alpha(\vec{r})) = i\vec{k}_\alpha \times (i\vec{k}_\alpha \times \vec{B}_\alpha) = -\left[\underbrace{(\vec{k}_\alpha\cdot\vec{B}_\alpha)}_{=0}\vec{k}_\alpha - (\vec{k}_\alpha\cdot\vec{k}_\alpha)\vec{B}_\alpha\right]$$

$$= |\vec{k}_\alpha|^2 \vec{B}_\alpha(\vec{r}) = \varepsilon_r \frac{\omega_\alpha^2}{c^2}\vec{B}_\alpha(\vec{r}) \qquad (10)$$

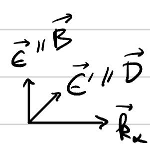
↳ Solutions of (10) for $|\vec{k}_\alpha| = \frac{\omega_\alpha}{c}\sqrt{\varepsilon_r}$

**Orthogonality**:
$$\int_V \vec{B}_\alpha^* \vec{B}_\beta \, d^3\vec{r} = C_\alpha^* C_\beta \, \vec{\epsilon}_\alpha^* \cdot \vec{\epsilon}_\beta \underbrace{\iiint_{-\frac{L}{2}}^{\frac{L}{2}} dx\,dy\,dz \; e^{i(\vec{k}_\beta - \vec{k}_\alpha)\cdot\vec{r}}}_{=0 \text{ if } \alpha\neq\beta}$$

**Normalized**:
$$\frac{1}{\mu_0}\int_V |\vec{B}_\alpha|^2 d^3\vec{r} = \frac{1}{\mu_0}|C_\alpha|^2 \int_V 1 \, d^3\vec{r} = \frac{1}{\mu_0} L^3 |C_\alpha|^2 = \frac{\hbar\omega_\alpha}{2}$$

$$\hookrightarrow \boxed{C_\alpha = \sqrt{\frac{\mu_0 \hbar\omega_\alpha}{2V}}} \qquad \left(c^2 = \frac{1}{\mu_0\varepsilon_0}\right)$$

From (11):
$$\vec{D}_\alpha = \frac{i}{\mu_0\omega_\alpha} i\vec{k}_\alpha \times \vec{B}_\alpha = \sqrt{\frac{\hbar\omega_\alpha}{\mu_0 2V}}\frac{1}{\omega_\alpha} e^{i\vec{k}_\alpha\cdot\vec{r}} \underbrace{(-\vec{k}_\alpha \times \vec{\epsilon}_\alpha)}_{\sqrt{\varepsilon_r}\frac{\omega_\alpha}{c}\vec{\epsilon}\,'_\alpha}$$




$$\boxed{\vec{D}_\alpha(\vec{r}) = \sqrt{\frac{\varepsilon_r\varepsilon_0\hbar\omega_\alpha}{2V}} e^{i\vec{k}_\alpha\cdot\vec{r}} \vec{\epsilon}\,'_\alpha}$$

modes = travelling plane waves

Finally, any field can be expressed as an operator:

$$\hat{\vec{D}}(\vec{r},t) = \sum_\alpha \sqrt{\frac{\varepsilon \hbar \omega_\alpha}{2V}} \left( \hat{a}_\alpha(t)\, e^{i\vec{k}_\alpha\cdot\vec{r}}\, \vec{\epsilon}_\alpha + h.c. \right)$$

and $$\hat{\vec{E}}(\vec{r},t) = \sum_\alpha \underbrace{\sqrt{\frac{\hbar\omega_\alpha}{2\varepsilon V}}}_{\mathcal{E}_\alpha} \left( \ldots \right) = \underbrace{\hat{\vec{E}}^{(+)}(\vec{r},t) + \hat{\vec{E}}^{(-)}(\vec{r},t)}_{\text{analytic signal}}$$

positive freq. ← $$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_\alpha \mathcal{E}_\alpha\, \hat{a}_\alpha(t)\, e^{i\vec{k}_\alpha\cdot\vec{r}}\, \vec{\epsilon}_\alpha$$

negative freq. ← $$\hat{\vec{E}}^{(-)}(\vec{r},t) = \sum_\alpha \mathcal{E}_\alpha\, \hat{a}^\dagger_\alpha(t)\, e^{-i\vec{k}_\alpha\cdot\vec{r}}\, \vec{\epsilon}^{\,*}_\alpha$$

Time evolution : Heisenberg equation

$$i\hbar \frac{d\hat{a}_\alpha}{dt} = [\hat{a}_\alpha, \hat{H}_L] = [\hat{a}_\alpha, \sum_\beta \hbar\omega_\beta\, \hat{a}^\dagger_\beta \hat{a}_\beta]$$

$$= \hbar\omega_\alpha \left( \hat{a}_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha - \hat{a}^\dagger_\alpha \hat{a}_\alpha \hat{a}_\alpha \right) = \hbar\omega_\alpha\, \hat{a}_\alpha$$

So $\frac{d\hat{a}_\alpha}{dt} = -i\omega_\alpha \hat{a}_\alpha \Rightarrow \boxed{\hat{a}_\alpha(t) = \hat{a}_\alpha(0)\, e^{-i\omega_\alpha t}}$ free evolution

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## Change of basis

For any unitary matrix $U$ $(U^\dagger = U^{-1})$ we can define a new basis of modes : $\vec{d}_\beta(\vec{r}) = \sum_\alpha U_{\beta\alpha} \vec{D}_\alpha(\vec{r})$ → orthonormal

and the operators tranform as $\hat{b}_\beta = \sum_\alpha U_{\beta\alpha}\, \hat{a}_\alpha$ and $\hat{b}^\dagger_\beta = \sum_\alpha (U^{-1})_{\alpha\beta}\, \hat{a}^\dagger_\alpha$

Commutation rules are conserved

$$[\hat{b}_\beta, \hat{b}^\dagger_{\beta'}] = \sum_\alpha \cancel{\sum_{\alpha'}} U_{\beta\alpha} (U^{-1})_{\alpha'\beta'} \underbrace{[\hat{a}_\alpha, \hat{a}^\dagger_{\alpha'}]}_{\delta_{\alpha\alpha'}} = \delta_{\beta\beta'}$$

Total photon number is conserved

$$\hat{N}^{(\beta)}_{tot} = \sum_\beta \hat{b}^\dagger_\beta \hat{b}_\beta = \sum_\beta \sum_\alpha \sum_{\alpha'} \underbrace{(U^{-1})_{\alpha\beta} U_{\beta\alpha'}}_{\delta_{\alpha\alpha'}} \hat{a}^\dagger_{\alpha'} \hat{a}_\alpha = \sum_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha = \hat{N}^{(\alpha)}_{tot}$$

But most other properties of " $\hat{a}^\dagger$ " depend on the chosen mode basis!

↳ (momentum, angular momentum, frequency ...)

Ex: standing wave in 1D (length L, fixed polarisation)

In the basis of travelling waves: $E_m(z) = \mathcal{E}_m e^{ik_m z}$ where

$k_m = m\frac{2\pi}{L}$ and $\mathcal{E}_m = \sqrt{\frac{\hbar\omega_m}{2\varepsilon L}}$

We can choose the unitary U that mixes modes with opposite $\vec{k}$ vectors.

standing waves

$$E_{m_c}(z) = \mathcal{E}_m \frac{1}{\sqrt{2}}\left(e^{+ik_m z} + e^{-ik_m z}\right) = \sqrt{2}\,\mathcal{E}_m \cos(k_m z)$$

$$E_{m_s}(z) = \mathcal{E}_m \frac{1}{\sqrt{2}}\left(e^{ik_m z} - e^{-ik_m z}\right) = i\sqrt{2}\,\mathcal{E}_m \sin(k_m z)$$

Create a photon in mode $\beta = m_c$ : $\hat{b}^\dagger_{m_c}|0\rangle = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{+m}|0\rangle + \hat{a}^\dagger_{-m}|0\rangle\right)$

↳ zero average momentum ($\frac{1}{2}$ probability $\pm k_m\hbar$)

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## Quantum theory of photo dection

Light - matter interaction with a two-level system in the dipole approximation:

$$\hat{H}_I(t) = -\frac{1}{\varepsilon_0}\,\hat{\vec{\mu}}\cdot\hat{\vec{D}}(\vec{r}_0, t)$$ at the position of the atom $\vec{r}_0 = \vec{0}$

↳ dipole operator