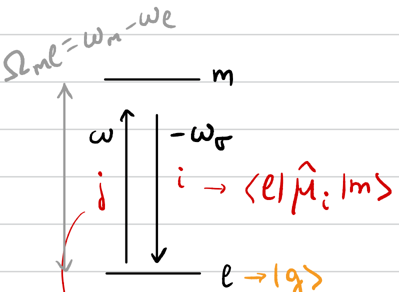
Week 05

I) Interpretation of the perturbative expressions

1) Linear suscept.

anti-resonant

$$\chi^{(1)}_{ij}(-\omega_\sigma:\omega) = \frac{N/V}{\varepsilon_0 \hbar} \sum_{\ell m} \rho^{(0)}_{\ell\ell} \left(\frac{\mu^i_{\ell m}\,\mu^j_{m\ell}}{\Omega_{m\ell} - \omega - i\gamma_{m\ell}} + \frac{\mu^j_{\ell m}\,\mu^i_{m\ell}}{\Omega_{m\ell} + \omega + i\gamma_{m\ell}} \right)$$



$\langle m | \hat{\mu}_j | \ell \rangle = \mu^j_{m\ell}$: $\ell \xrightarrow{j} m$

- upward arrow : photon annihilation; positive frequency
- downward — photon creation; negative frequency; amplification / stimulated emission

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2) Perturbative expression for $\chi^{(2)}$

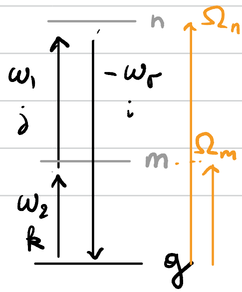
* We consider all the population in the ground state

* We look at sum-frequency generation (SFG) : $\omega_\sigma = \omega_1 + \omega_2$, $\omega_1, \omega_2 > 0$

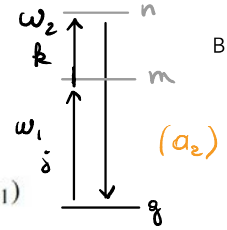
Only two terms can become near-resonant :

$$\chi^{(2)}_{ijk}(-\omega_\sigma;\omega_1,\omega_2) = \frac{N/V}{2\varepsilon_0\hbar^2} \sum_{mn} \left\{ \frac{\mu^i_{gn}\,\mu^j_{nm}\,\mu^k_{mg}}{(\Omega_n - \omega_\sigma)(\Omega_m - \omega_2)} \right. \quad (a_1) \rightarrow$$

$$+ \frac{\mu^i_{gn}\,\mu^k_{nm}\,\mu^j_{mg}}{(\Omega_n - \omega_\sigma)(\Omega_m - \omega_1)} \quad (a_2)$$



General expression for $\chi^{(2)}$



Boyd Ch. 3

$$\chi^{(2)}_{ijk}(\omega_p+\omega_q, \overset{\omega_1}{\omega_q}, \overset{\omega_2}{\omega_p}) \qquad \omega_\sigma = \omega_1 + \omega_2$$

$$
\begin{aligned}
&= \frac{N}{2\epsilon_0\hbar^2}\sum_{lmn}\rho_{ll}^{(0)}\left\{\frac{\mu^i_{ln}\mu^j_{nm}\mu^k_{ml}}{[(\omega_{nl}-\omega_p-\omega_q)-i\gamma_{nl}][(\omega_{ml}-\omega_p)-i\gamma_{ml}]}\right. && (a_1)\\
&+\frac{\mu^i_{ln}\mu^k_{nm}\mu^j_{ml}}{[(\omega_{nl}-\omega_p-\omega_q)-i\gamma_{nl}][(\omega_{ml}-\omega_q)-i\gamma_{ml}]} && (a_2)\\
&+\frac{\mu^k_{ln}\mu^i_{nm}\mu^j_{ml}}{[(\omega_{mn}-\omega_p-\omega_q)-i\gamma_{mn}][(\omega_{nl}+\omega_p)+i\gamma_{nl}]} && (a_1')\\
&+\frac{\mu^j_{ln}\mu^i_{nm}\mu^k_{ml}}{[(\omega_{mn}-\omega_p-\omega_q)-i\gamma_{mn}][(\omega_{nl}+\omega_q)+i\gamma_{nl}]} && (a_2')\\
&+\frac{\mu^j_{ln}\mu^i_{nm}\mu^k_{ml}}{[(\omega_{nm}+\omega_p+\omega_q)+i\gamma_{nm}][(\omega_{ml}-\omega_p)-i\gamma_{ml}]} && (b_1)\\
&+\frac{\mu^k_{ln}\mu^i_{nm}\mu^j_{ml}}{[(\omega_{nm}+\omega_p+\omega_q)+i\gamma_{nm}][(\omega_{ml}-\omega_q)-i\gamma_{ml}]} && (b_2)\\
&+\frac{\mu^k_{ln}\mu^j_{nm}\mu^i_{ml}}{[(\omega_{ml}+\omega_p+\omega_q)+i\gamma_{ml}][(\omega_{nl}+\omega_p)+i\gamma_{nl}]} && (b_1')\\
&+\left.\frac{\mu^j_{ln}\mu^k_{nm}\mu^i_{ml}}{[(\omega_{ml}+\omega_p+\omega_q)+i\gamma_{ml}][(\omega_{nl}+\omega_q)+i\gamma_{nl}]}\right\}. && (b_2')
\end{aligned}
$$

far off-resonance (lossless)

DFG

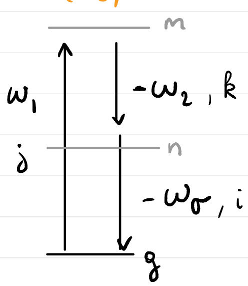
$(a_1') + (b_2)$: $$\boxed{\frac{\mu^k_{ln}\mu^i_{nm}\mu^j_{ml}}{(\omega_{nl}+\omega_p)(\omega_{ml}-\omega_q)}}$$

$(a_2') + (b_1)$: $$\frac{\mu^j_{ln}\mu^i_{nm}\mu^k_{ml}}{(\omega_{nl}+\omega_q)(\omega_{ml}-\omega_p)}$$

* <u>For DFG</u> $\omega_1 > \omega_2 > 0$ $\omega_\sigma = \omega_1 - \omega_2 > 0$

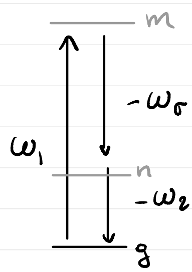
$\chi^{(2)}_{ijk}(-\omega_\sigma : \omega_1, -\omega_2) \rightarrow$ Two terms : (a_2) and $(a_1') + (b_2)$

(a_2)



$\mu^i_{gn}\mu^k_{nm}\mu^j_{mg}$

$(a_1') + (b_2)$



II) Wave propagation in a non linear medium

$$\vec{\nabla}\times\vec{E} = -\mu_0 \dot{\vec{H}}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon_0 \dot{\vec{E}} + \dot{\vec{P}} \quad \rightarrow \quad \vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\frac{1}{c^2}\ddot{\vec{E}} - \frac{1}{\varepsilon_0 c^2}\ddot{\vec{P}}(\vec{E})$$

1) We neglect beam walk-off : $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E}$

2) We expand $\vec{P}(\vec{E}) = \vec{P}^{(1)}(\vec{E}) + \vec{P}^{(2)}(\vec{E}) \dots$

$$-\nabla^2\vec{E}(\vec{r},t) + \frac{1}{c^2}\ddot{\vec{E}}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\ddot{\vec{P}}^{(1)}(\vec{r},t) = -\frac{1}{\varepsilon_0 c^2}\ddot{\vec{P}}^{(n)}(\vec{r},t)$$

Fourier transform $t \rightarrow \omega$:

$$(1) \quad \boxed{\nabla^2 \underbrace{\vec{E}_\alpha}_{\vec{E}(\omega_\alpha)}(\vec{r}) + \overbrace{\boxed{\frac{\omega_\alpha^2\left(1+\chi_\alpha^{(1)}\right)}{c^2}}}^{k_\alpha^2}\vec{E}_\alpha(\vec{r}) = \frac{-\omega_\alpha^2}{\varepsilon_0 c^2}\vec{P}_\alpha^{(n)}(\vec{r})}$$

$$\vec{P}_\alpha^{(1)} = \varepsilon_0 \chi_\alpha^{(1)} E_\alpha$$

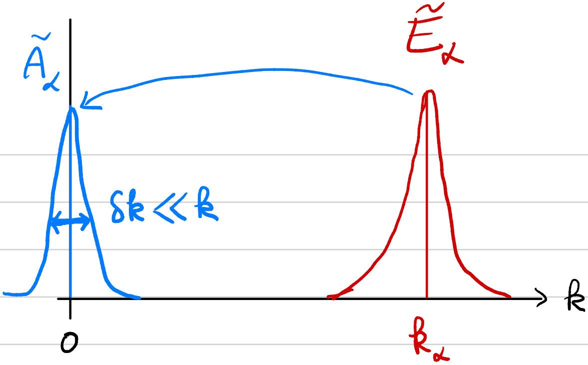
3) We consider plane waves polarized along $\vec{u}_\alpha$ propagating along z.

4) Slowly varying envelope approximation (SVEA):

$$\boxed{\vec{E}_\alpha(z) = A_\alpha(z)\,\vec{u}_\alpha\, e^{ik_\alpha z}} \quad \text{with} \quad k_\alpha = \frac{\omega_\alpha n_\alpha}{c}$$

$$n_\alpha = \sqrt{1+\chi_\alpha^{(1)}}$$

$$\tilde{E}_\alpha(k)$$



$$\frac{\partial \tilde{A}_\alpha}{\partial z} \approx |\delta k\, \tilde{A}_\alpha| \ll k_\alpha A_\alpha$$

$$\boxed{\frac{\partial^2 A_\alpha}{\partial z^2} \ll k_\alpha \frac{\partial A_\alpha}{\partial z}}$$

SVEA

Now: $$\nabla^2 \vec{E}_\alpha(\vec{r}) = \Big(\cancel{\frac{\partial^2 A_\alpha}{\partial z^2}} + 2ik_\alpha \frac{\partial A_\alpha}{\partial z} \underbrace{- k_\alpha^2 A_\alpha \Big) e^{ik_\alpha z}\, \vec{u}_\alpha}_{\text{inject in (1) : cancel with } k_\alpha \vec{E}_\alpha(\vec{r})}$$

$$\hookrightarrow \quad \boxed{\frac{\partial A_\alpha}{\partial z} = \frac{i\omega_\alpha c}{2\varepsilon_0 n_\alpha}\, \vec{P}_\alpha^{(n)}(z)\cdot\vec{u}_\alpha\, e^{-ik_\alpha z}} \quad (2)$$

$\vec{E}(-\omega_n) = \vec{E}_n^*$

where : $$\vec{P}_\alpha^{(n)} = \varepsilon_0 K\, \chi^{(n)}(-\omega_\alpha : \omega_1, \dots, \omega_n) : \vec{E}(\omega_1)\dots\vec{E}(\omega_n)$$

$$= \varepsilon_0 K \left(\chi^{(n)}(\dots) : \vec{u}_1 \dots \vec{u}_n\right) A(\omega_1)\dots A(\omega_n)\, e^{i(k_1 + \dots k_n)z}$$

$$\boxed{\frac{\partial A_\alpha}{\partial z} = \frac{i\omega_\alpha c}{2 n_\alpha}\, \chi_{eff}^{(n)}\, A_1^{(*)}(z) \dots A_n^{(*)}(z)\, e^{i\Delta k z}}$$

where
* $\Delta k = k_1 + \dots + k_n - k_\alpha = \frac{1}{c}(n_1\omega_1 + \dots + n_n\omega_n - n_\alpha\omega_\alpha)$

phase mismatch

* $\chi_{eff}^{(n)} = K\, \vec{u}_\alpha \cdot \chi^{(n)}(\dots) : \vec{u}_1 \dots \vec{u}_n$

effective nonlinear suscept.