Microscopic Maxwell's equations

We consider the microscopic fields $\boldsymbol{e}(\boldsymbol{r}, t)$ and $\boldsymbol{b}(\boldsymbol{r}, t)$ existing in vacuum, together with a distribution of charges and currents described by $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{j}(\boldsymbol{r}, t)$.
Maxwell's equations are:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{b} &= 0 && \text{no magnetic monopole} && (1)\\
\boldsymbol{\nabla} \times \boldsymbol{e} + \frac{\partial \boldsymbol{b}}{\partial t} &= 0 && \text{Faraday's law of induction} && (2)\\
\boldsymbol{\nabla} \times \left(\frac{1}{\mu_0}\boldsymbol{b}\right) - \frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{e}) &= \boldsymbol{j} && \text{Ampere's law with displ. current} && (3)\\
\boldsymbol{\nabla} \cdot (\varepsilon_0 \boldsymbol{e}) &= \rho && \text{Gauss' law} && (4)
\end{aligned}$$

We see that eqs. (3) and (4) contain the source terms.

$$\dot{\rho} = \varepsilon_0 \vec{\nabla} \cdot \left(\frac{\partial \vec{e}}{\partial t}\right) = \vec{\nabla} \cdot \left(-\vec{j} + \vec{\nabla} \times \frac{\vec{b}}{\mu_0}\right) = -\vec{\nabla} \cdot \vec{j}$$

$\frac{\partial}{\partial t}(4)$ $\qquad$ $\nabla \cdot (3)$

Reminder: Fourier Transform

We define the direct and inverse Fourier transforms as follows:

$$\boldsymbol{e}(\boldsymbol{r},t) = \iiint_{-\infty}^{+\infty} dk^3 \int_{-\infty}^{+\infty} d\omega\, \boldsymbol{e}(\boldsymbol{k},\omega) \exp[i(\boldsymbol{k}\cdot r - \omega t)] \quad (5)$$

$$\boldsymbol{e}(\boldsymbol{k},\omega) = \iiint_{-\infty}^{+\infty} \frac{dr^3}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dt}{2\pi}\, \boldsymbol{e}(\boldsymbol{r},t) \exp[-i(\boldsymbol{k}\cdot r - \omega t)] \quad (6)$$

Properties of the derivation:

$$\frac{\partial}{\partial t}\boldsymbol{e}(\boldsymbol{r},t) \longleftrightarrow -i\omega\boldsymbol{e}(\boldsymbol{k},\omega) \quad (7)$$

$$\frac{\partial}{\partial x}\boldsymbol{e}(\boldsymbol{r},t) \longleftrightarrow ik_x\boldsymbol{e}(\boldsymbol{k},\omega) \quad (8)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{e}(\boldsymbol{r},t) \longleftrightarrow i\boldsymbol{k}\cdot\boldsymbol{e}(\boldsymbol{k},\omega) \quad (9)$$

$$\boldsymbol{\nabla}\times\boldsymbol{e}(\boldsymbol{r},t) \longleftrightarrow i\boldsymbol{k}\times\boldsymbol{e}(\boldsymbol{k},\omega) \quad (10)$$

Longitudinal and transverse fields

Helmholtz decomposition

Any physically relevant field $\boldsymbol{A}$ can be decomposed as a sum $\boldsymbol{A} = \boldsymbol{A}_{\parallel} + \boldsymbol{A}_{\perp}$ where

- $\boldsymbol{A}_{\parallel}$ is the **longitudinal field** satisfying $\boldsymbol{A}_{\parallel}(\boldsymbol{k},\omega) \parallel \boldsymbol{k}$, or $\boldsymbol{\nabla} \times \boldsymbol{A}_{\parallel}(\boldsymbol{r},t) = 0$
- $\boldsymbol{A}_{\perp}$ is the **transverse field** satisfying $\boldsymbol{A}_{\perp}(\boldsymbol{k},\omega) \perp \boldsymbol{k}$, or $\boldsymbol{\nabla} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r},t) = 0$

Remark

For any scalar field f and vectorial field $\boldsymbol{A}$ we have the identities

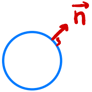
- $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$ (The longitudinal field is the gradient of a scalar potential)
- $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ (The transverse field is the curl of a vector potential)

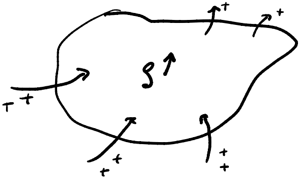
Properties of the electric field

- A **static** electric field is purely **longitudinal** (eq. 2)
- The **radiated** field in vacuum is purely **transverse** (eq. 4)

Charge conservation

$$\dot{\rho} = -\vec{\nabla}\cdot\vec{j}$$





Reminder :

$$\iiint_V \vec{\nabla}\cdot\vec{j}\, dV = \oiint_S \vec{j}\cdot\vec{n}\, d\vec{n}$$

Wave equation in vacuum without sources ($\boldsymbol{j}$ and ρ zero)

Let's first define the **vector Laplacian**

$$\nabla^2 \boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = (\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z)$$

where the Laplacian is $\nabla^2 f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$. Taking the curl of eq. (2) and using eqs. (3,4) we obtain

$$0 = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{e}) + \varepsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{e}}{\partial t^2} \quad (11)$$

$$= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{e}) - \nabla^2 \boldsymbol{e} + \varepsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{e}}{\partial t^2} \quad (12)$$

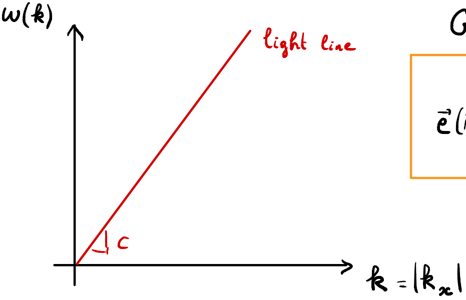
$$\boxed{0 = \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \boldsymbol{e}} \quad (13)$$

We defined $\frac{1}{c^2} = \varepsilon_0 \mu_0$, the electromagnetic wave velocity in vacuum. Eq. (13) is the wave equation for $\boldsymbol{e}$.

Dispersion relation

(13) → FT : $0 = \left(-|\vec{k}|^2 + \frac{\omega^2}{c^2}\right)\vec{e}(\vec{k},\omega) \quad \forall\, \vec{k},\omega$

$$\Rightarrow \quad |\vec{k}|^2 = \frac{\omega^2}{c^2} = \varepsilon_0\mu_0\omega^2$$



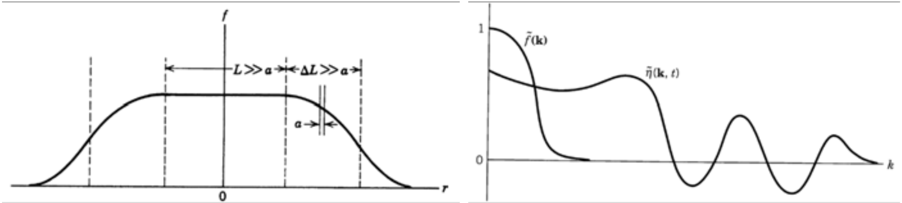
General solution of (13)

$$\vec{e}(\vec{r},t) = \iiint \frac{dk^3}{(2\pi)^3} A_{\vec{k}}\, \varepsilon_{\vec{k}}\, e^{-i(\vec{k}\cdot\vec{r} - \omega(k)t)}$$

$A_{\vec{k}}\,\varepsilon_{\vec{k}}$: amplitude and polarisation of plane waves

Macroscopic Maxwell's equations

The number of microscopic charges and currents in a medium is untractable. We will average the microscopic fields over a length scale L much larger than the interatomic distance a but much smaller than the optical wavelength λ. We introduce a normalized smoothing function $f(\boldsymbol{r})$ that varies slowly over distances on the order of a, but whose support is much smaller than λ.



The shape of a smoothing function in real and reciprocal space. $\eta(\boldsymbol{k})$ illustrates the behavior of a microscopic field [1].

Definition of the macroscopic fields

$\boldsymbol{E}(\boldsymbol{r}, t) = \iiint_{-\infty}^{+\infty} du^3\, f(\boldsymbol{u})\, \boldsymbol{e}(\boldsymbol{r} - \boldsymbol{u}, t) = \langle e(\boldsymbol{r}, t) \rangle$ (average over a ball of radius L)

[1] cf. Jackson (Ch. 6.6) and Vanderlinde (Ch. 8.1)

Macroscopic Maxwell's equations

All partial derivatives of the macroscopic fields satisfy $\frac{\partial \boldsymbol{E}}{\partial x} = \langle \frac{\partial \boldsymbol{e}(\boldsymbol{r},t)}{\partial x} \rangle$, etc., and by the linearity of Maxwell's equations we obtain the macroscopic version:

$$\nabla \cdot \boldsymbol{B} = 0 \quad (14)$$

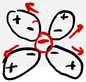
$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad (15)$$

$$\nabla \times \left(\frac{1}{\mu_0} \boldsymbol{B} \right) - \frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{E}) = \langle \boldsymbol{j}(\boldsymbol{r}, t) \rangle \quad (16)$$

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E}) = \langle \rho(\boldsymbol{r}, t) \rangle \quad (17)$$

Then, we distinguish the induced charges and currents (due to the 'internal' or 'intrinsic' charges in the medium) $\rho_{\rm ind}$ and $\boldsymbol{j}_{\rm ind}$, from the external charges and currents controlled by an independent mean, $\rho_{\rm ext}$ and $\boldsymbol{j}_{\rm ext}$. In this lecture, unless stated otherwise, we will always consider the latter to be zero.

Macroscopic Maxwell's equations



From Problem Set 01, we introduce the induced polarisation density inside the medium $\boldsymbol{P}(\boldsymbol{r}, t)$ satisfying $\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\rho_{\mathrm{ind}}$. From eq. (17) we get

$$\boldsymbol{\nabla} \cdot (\varepsilon_0 \boldsymbol{E}) = -\boldsymbol{\nabla} \cdot (\boldsymbol{P}) \iff \boldsymbol{\nabla} \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = 0 \iff \boldsymbol{\nabla} \cdot \boldsymbol{D} = 0$$

where we defined the **electric displacement field** $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
The **macroscopic Maxwell's equations** eventually writes

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 & \qquad \boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{D} &= 0 & \qquad \boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} &= 0 \end{aligned}$$

together with **two constitutive equations** for the medium

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}(\boldsymbol{E}) \qquad (18)$$

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \approx \mu_0 \boldsymbol{H} \qquad \text{(no magnetic response)} \qquad (19)$$

Remark : induced currents

Charge conservation : $\dot{\rho}_{ind} + \vec{\nabla} \cdot \vec{j}_{ind} = 0$

$$-\vec{\nabla} \cdot \dot{\vec{P}} + \vec{\nabla} \cdot \vec{j}_{ind} = 0$$

$$\vec{\nabla} \cdot \left(\vec{j}_{ind} - \dot{\vec{P}} \right) = 0 \qquad \vec{j}_{ind} \neq \frac{\partial \vec{P}}{\partial t}$$

But for transverse fields: $\vec{j}_{ind,\perp} = \dfrac{\partial \vec{P}_\perp}{\partial t}$

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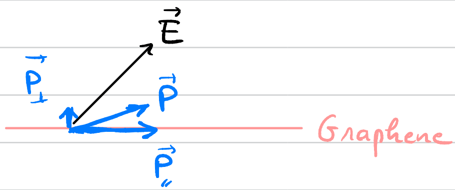
* Linear static (d.c.) susceptibility



$$\vec{P} = \varepsilon_0 \underline{\underline{\chi}}^{(1)} \cdot \vec{E}$$

↳ rank - 2 tensor

$$P_i = \varepsilon_0 \chi^{(1)}_{ij} E_j = \varepsilon_0 \sum_{j=x,y,z} \chi^{(1)}_{ij} E_j$$

$i = x, y, z$ Sum over j

If the material is reciprocal (time - reversal symmetry)

$\underline{\underline{\chi}}^{(1)}$ is real and symmetric → we can diagonalize it

⇒ 3 principal axes (optics axes) x, y, z

$$\underline{\underline{\chi}}^{(1)} = \begin{pmatrix} \chi_{xx} & 0 & 0 \\ 0 & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix}$$

A material is isotropic if $\chi_{xx} = \chi_{yy} = \chi_{zz}$ { amorphous, cubic crystals, liquids, gases

Otherwise it is anisotropic.

* Linear optical susceptibility

May I write ~~$\vec{P}(\vec{r},t) = \varepsilon_0 \underline{\underline{\chi}}^{(1)} \cdot \vec{E}(\vec{r},t)$~~ ?

↳ instantaneous response !

For Homogeneous, Time-translation invariant medium:

$$\vec{P}(\vec{r},t) = \varepsilon_0 \iiint du^3 \int d\tau \; \underline{\underline{R}}^{(1)}(\vec{u},\tau)\,\vec{E}(\vec{r}-\vec{u}, t-\tau)$$

"non-local response" ⇒ neglected

$$\vec{P}(t) = \varepsilon_0 \int_{-\infty}^{+\infty} d\tau \; \underline{\underline{R}}^{(1)}(\tau)\,\vec{E}(t-\tau)$$

FT:

$$\vec{P}(\omega) = \varepsilon_0 \int_{-\infty}^{+\infty} d\tau \int dt \; R^{(1)}(\tau)\,\vec{E}(t-\tau)\, e^{-i\omega(t-\tau+\tau)}$$

↳ $d(t-\tau) = dt$, du

$$= \varepsilon_0 \underbrace{\int d\tau \; R^{(1)}(\tau)\, e^{-i\omega\tau}} \int du \; \vec{E}(u)\, e^{-i\omega u}$$

$$= \varepsilon_0 \, \underline{\underline{\chi}}^{(1)}(\omega)\, \vec{E}(\omega)$$

$$\vec{P}(\omega) = \varepsilon_0 \, \underline{\underline{\chi}}^{(1)}(\omega) \cdot \vec{E}(\omega)$$

* Dispersion relation

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{D}(\omega) = \varepsilon_0 \left(\underline{\underline{\mathbb{I}}} + \underline{\underline{\chi}}^{(1)}(\omega) \right) \cdot \vec{E}(\omega)$$

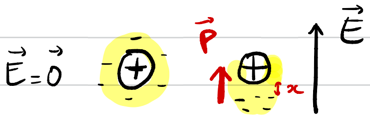
Along a principal axis 'x': $D_x(\omega) = \varepsilon_0 \big(\underbrace{1 + \chi_{xx}(\omega)} \big) E_x(\omega)$

$\varepsilon_r(\omega)$ relative permittivity

Phase velocity: $|\vec{k}|^2 = \mu_0 \varepsilon_0 \varepsilon_r(\omega)\, \omega^2 = \frac{\omega^2}{c^2} n^2(\omega)$

Refractive index: $\boxed{n(\omega) = \sqrt{1 + \chi(\omega)} = \sqrt{\varepsilon_r(\omega)}}$ $\in \mathbb{C}$

III) Lorentz oscillator model



$\vec{p} = \underset{[m^{-3}]}{N} \, q \, x \, \vec{e}_x$ for $\vec{E}$ along x

Newton's law : $m\ddot{x} = -m\omega_0^2 x - m\gamma_0 \dot{x} + qE(t)$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$

(m → reduced mass; $-m\omega_0^2 x$ → restoring; $-m\gamma_0\dot{x}$ → damping; $qE(t)$ → drive)

* **Harmonic solutions**

$$\begin{cases} x(t) = A(\omega)\, e^{-i\omega t} & \text{response} \\ E(t) = E(\omega)\, e^{-i\omega t} & \to \text{excitation} \end{cases}$$

Eq of motion : $(-\omega^2 + \omega_0^2 - i\gamma_0\omega)\, A(\omega) = \frac{q}{m} E(\omega)$

$$P(\omega) = Nq\, A(\omega) = \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma_0} E(\omega) \equiv \varepsilon_0\, \chi(\omega)\, E(\omega)$$

$$\chi(\omega) = \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\gamma_0\omega} \quad \text{with} \quad \chi_0 = \frac{Nq^2}{\varepsilon_0 m \omega_0^2}$$

Total response :

$$\boxed{\chi^{(1)}(\omega) = \sum_\alpha \chi_\alpha \frac{\omega_\alpha^2}{\omega_\alpha^2 - \omega^2 - i\gamma_\alpha\omega}}$$

↳ each d.o.f is characterized by 3 parameters

χ_α = strength $\qquad \omega_\alpha$ = resonance freq. $\qquad \gamma_\alpha$ = damping

* ## Sellmeier equations

Far from resonance χ is real : $\chi_\alpha = \frac{\omega_\alpha^2}{\omega_\alpha^2 - \omega} \chi_\alpha$

$$= \frac{1/\lambda_\alpha^2}{1/\lambda_\alpha^2 - 1/\lambda^2} \chi_\alpha = \frac{\lambda^2}{\lambda^2 - \lambda_\alpha^2} \chi_\alpha$$

In the transparency domain of a material we can write its refractive index :

$$\boxed{n^2 = 1 + \chi(\omega) = 1 + \sum_\alpha \frac{\lambda^2}{\lambda^2 - \lambda_\alpha^2} \chi_\alpha}$$

Sellmeier coefficient : λ_α , χ_α

* ## Complex index of refraction

$$\tilde{n}(\omega) = \sqrt{1 + \chi(\omega)} = n + i\kappa \quad \text{with} \quad \kappa = \mathrm{Im}\{\tilde{n}\}$$

Consider a plane wave satisfying the dispersion relation :

$$\tilde{k}(\omega) = \tilde{n}\frac{\omega}{c} = n\frac{\omega}{c} + i\kappa\frac{\omega}{c}$$

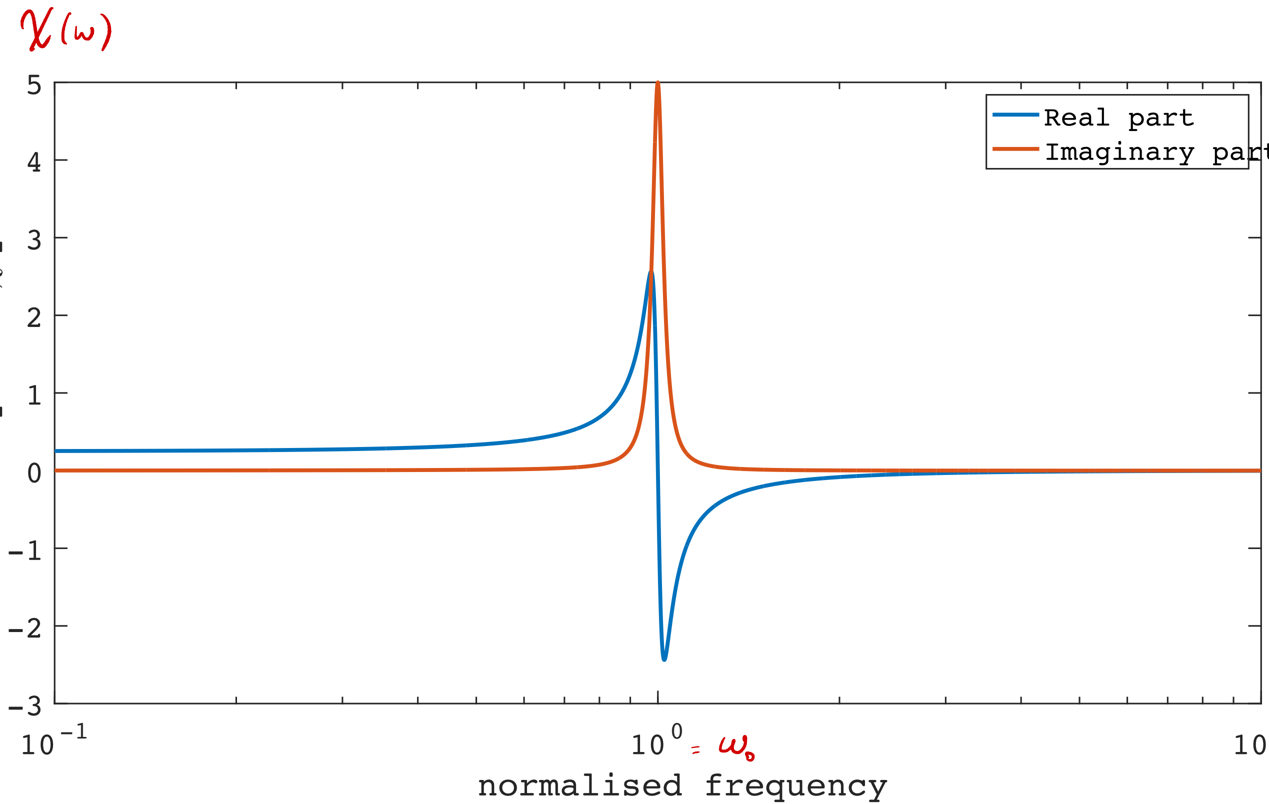
$$E_x(z,t) = E_0 e^{i(\tilde{k}z - \omega t)} = E_0 e^{i(kz - \omega t)} \underbrace{e^{-\kappa\frac{\omega}{c}z}}_{\text{damping of amplitude}}$$

Intensity $\propto |E|^2 = E_0^2 e^{-\frac{2\omega}{c}\kappa z}$ $\longrightarrow$ absorption coefficient

$$\alpha = \frac{2\omega}{c}\kappa = \frac{4\pi}{\lambda}\kappa = \frac{\omega}{nc}\mathrm{Im}\chi$$

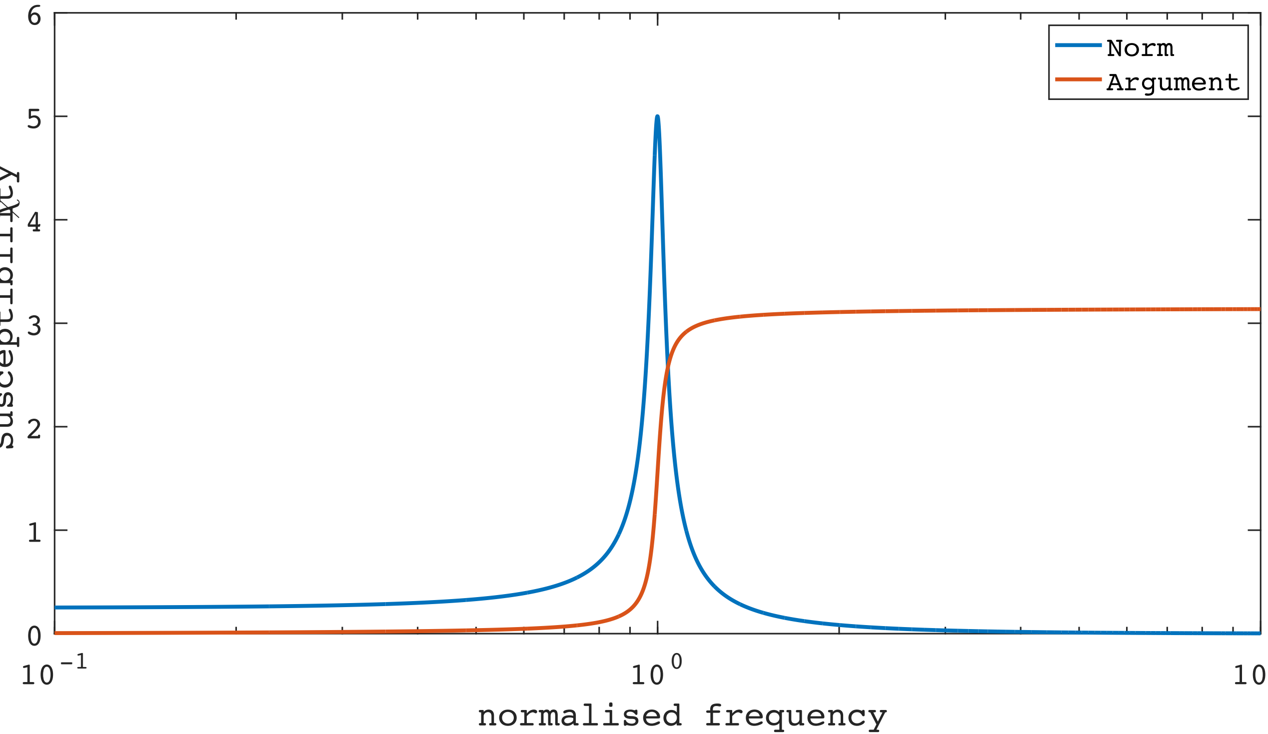
$$\alpha = \frac{2\omega}{c}\kappa = \frac{4\pi}{\lambda}\kappa \quad [m^{-1}]$$

↳ α^{-1} = propagation length $= \frac{\lambda}{4\pi\kappa}$ → if $\kappa = 0.1$, $\alpha^{-1} < \lambda$

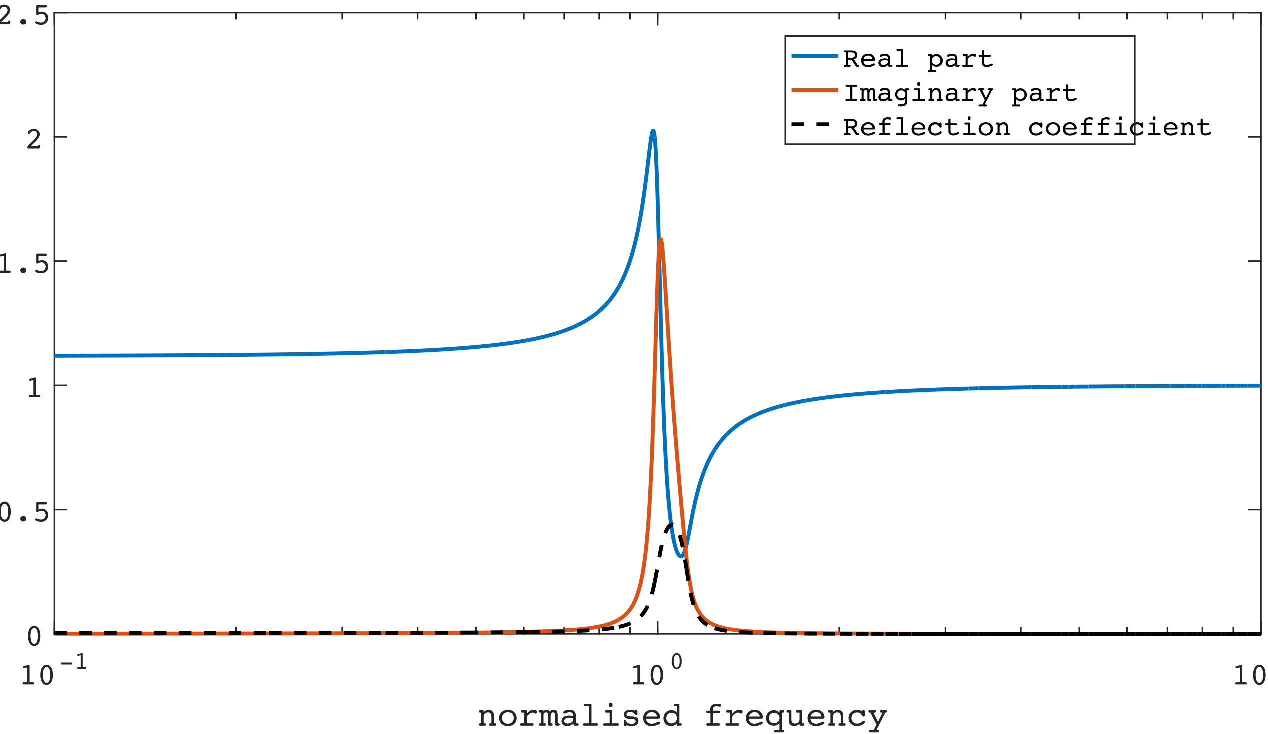


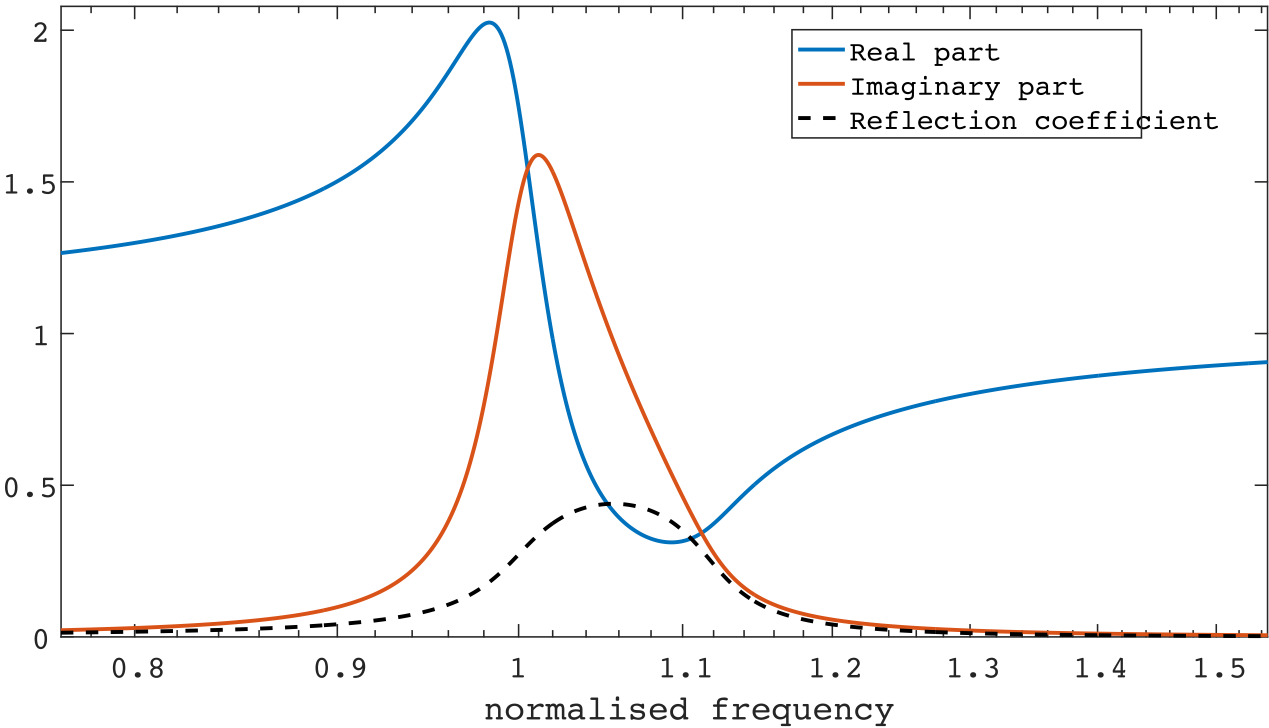
χ(ω)
5
4
3
2
1
0
−1
−2
−3
Real part
Imaginary part
10 −1
10 0 = ω0
10
normalised frequency

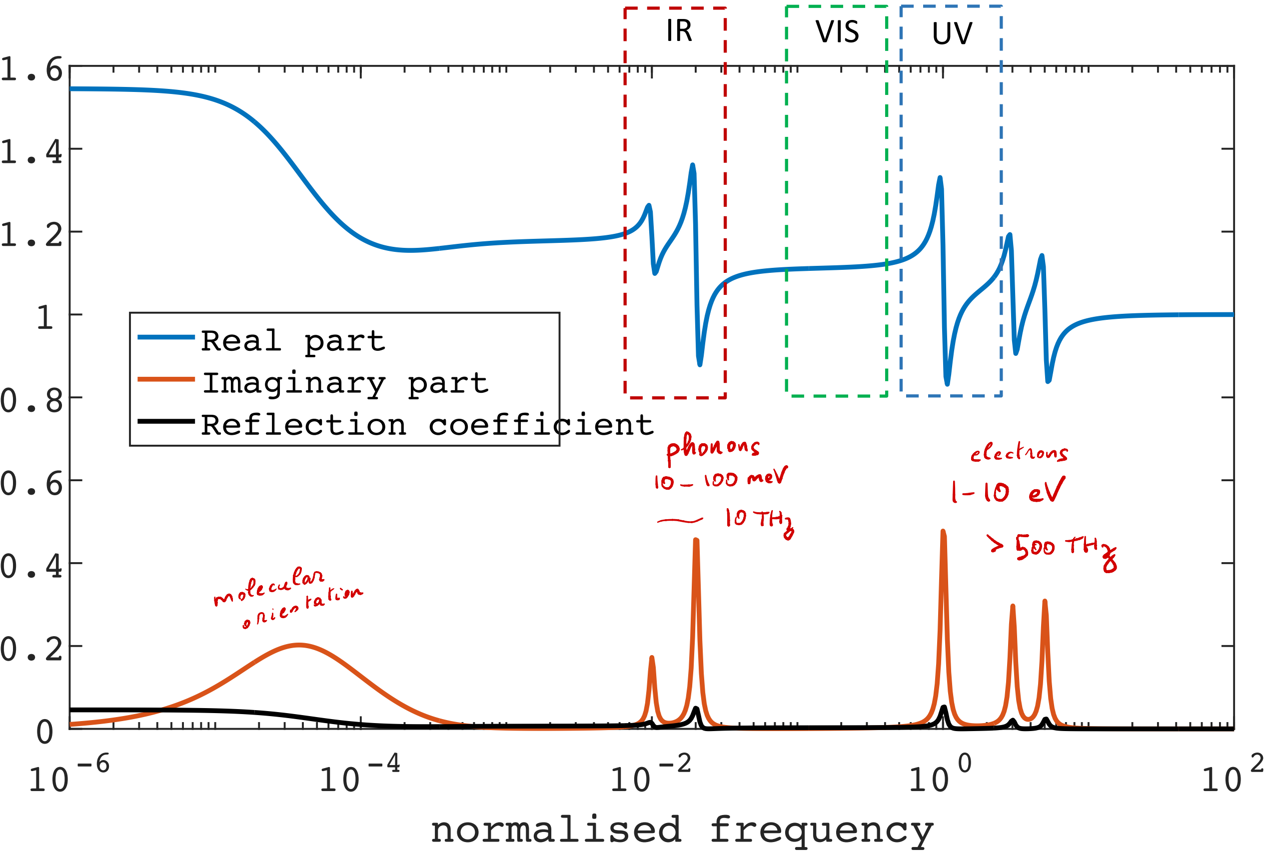
$\chi(\omega)$



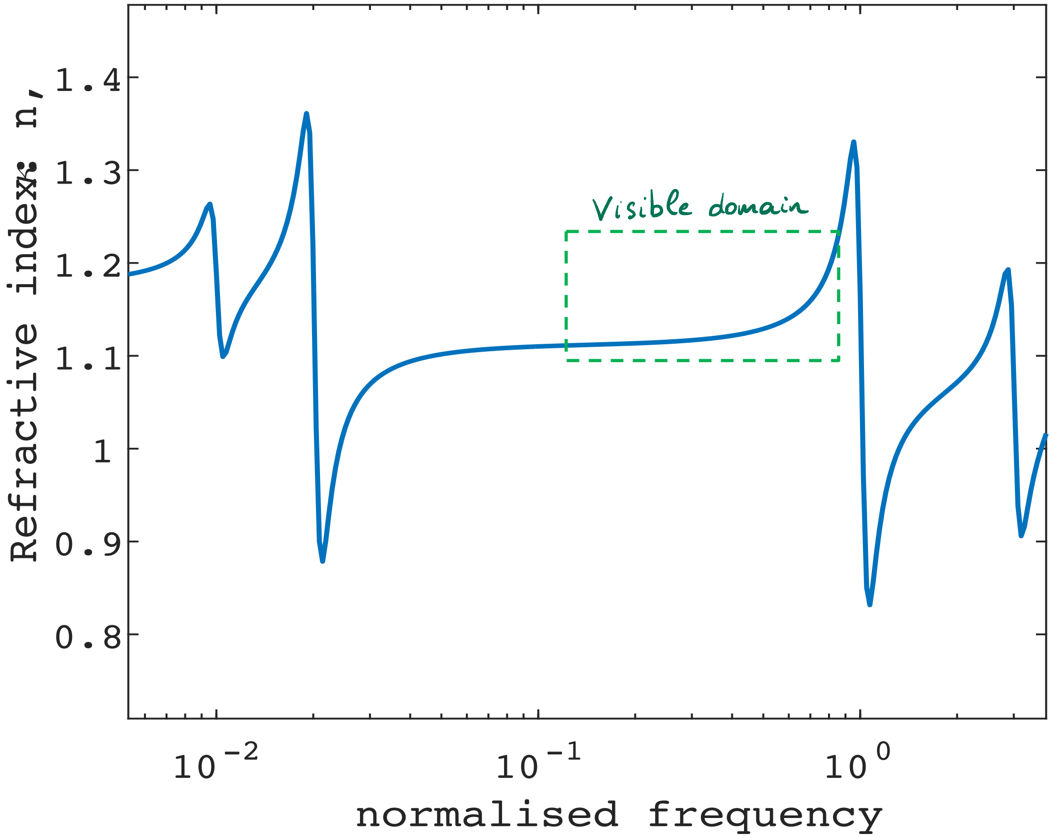
6
5
4
3
2
1
0
susceptibility
Norm
Argument
10^-1
10^0
10
normalised frequency

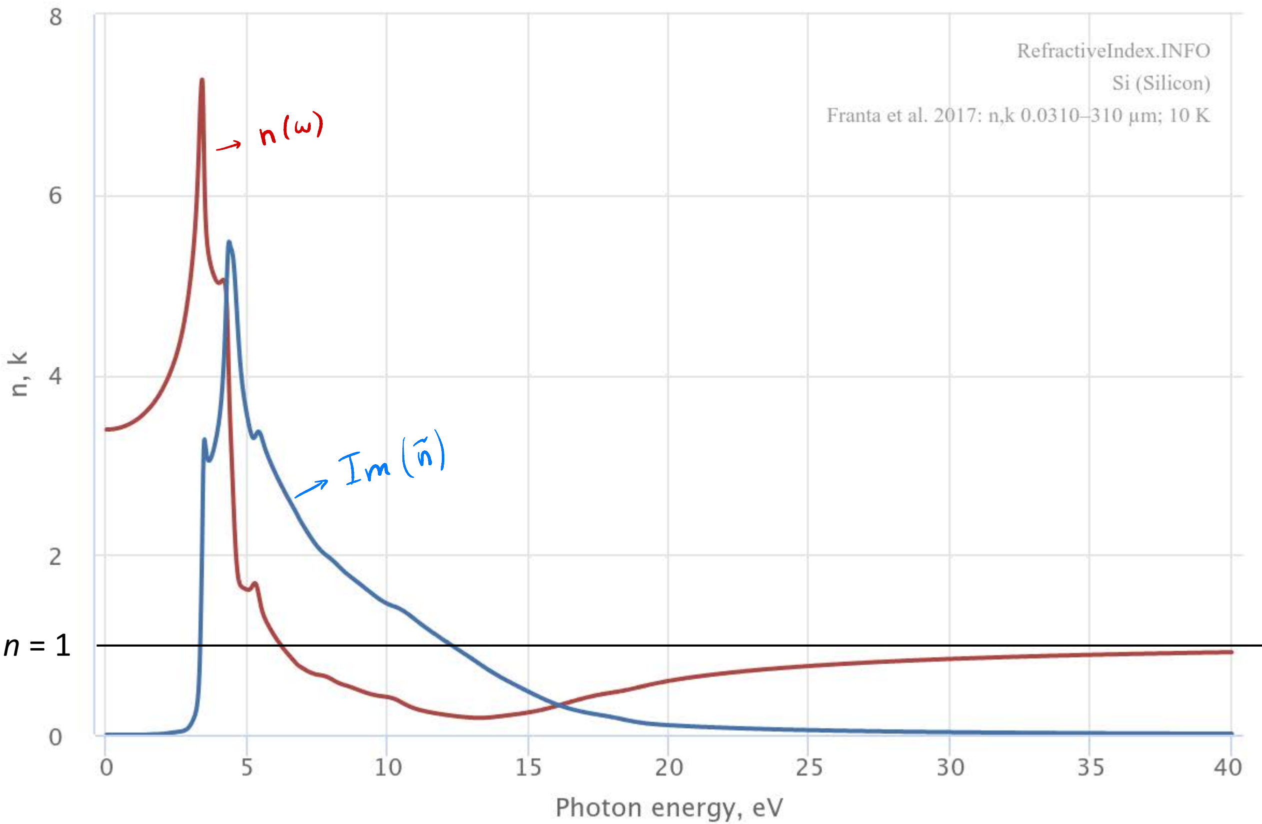

Real part
Imaginary part
Reflection coefficient
2.5
2
1.5
1
0.5
0
10^{-1}
10^{0}
10
normalised frequency


2
1.5
1
0.5
0
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
normalised frequency
Real part
Imaginary part
Reflection coefficient



$$n(\omega) = \mathrm{Re}\sqrt{1+\chi(\omega)} \quad \alpha(\omega) \approx \frac{2\omega}{c}\mathrm{Im}\sqrt{1+\chi(\omega)}$$


Visible domain
Refractive index: n, κ
1.4
1.3
1.2
1.1
1
0.9
0.8
10^{-2}
10^{-1}
10^{0}
normalised frequency



https://refractiveindex.info/?shelf=main&book=Si&page=Franta-10K