# Solutions to exercice sheet 5

Construction of Wiener process

**1.** Find a basis and a sub-basis for the canonical topology on $\mathbb{R}$.

Consider the set

$$\beta := \{]a,\infty[ : \ a \in \mathbb{R}\} \cup \{]-\infty, a] \ : \ a \in \mathbb{R}\}.$$

Then $\beta$ is a collection of open sets on $\mathbb{R}$ and intersections of elements in $\beta$ yield all open intervals on $\mathbb{R}$. But this last collection is a basis for the Euclidean topology on $\mathbb{R}$, so that $\beta$ is a basis for this topology.

**2.** Let $(\Omega, \tau)$ be a topological space. Let $\mathcal{S} \subset \mathcal{P}(\tau)$ be a totally ordered set (for the partial order being the set inclusion) so that no element of $\mathcal{S}$ has a finite sub-cover for $\Omega$. Show then that $\cup_{S\in\mathcal{S}}$ has no finite sub-cover of $\Omega$.
For $k = 1, \ldots, n$ let $\mathcal{M}_k$ be a finite collection of open sets of some topological space $(\Omega, \tau)$. For each $k$, Let $B_k \in \tau$ so that $\mathcal{M}_k \cup \{B_k\}$ covers $\Omega$. Show then that $(\cup_{k=1}^n \mathcal{M}_k) \cup \{\cap_{k=1}^n B_k\}$ covers $\Omega$.

If $\cup_{S\in\mathcal{S}}$ had a finite sub-cover of $\Omega$, then there would be some finite set $\{v_1, \ldots, V_n\} \subset \cup_{S\in\mathcal{S}}$, so that $\Omega = \cup_{k=1}^n V_n$. But then, there would be sets $\{S_{j_1}, \ldots, S_{j_n}\} \subset \mathcal{S}$ with $\forall k = 1, \ldots, n$, $V_k \in S_{j_k}$. Since $\mathcal{S}$ is totally ordered, there must be a set $T \in \mathcal{S}$ with $T \supset \cup_{k=1}^n S_{j_k}$. But then, $\forall k = 1, \ldots, n$, we would have $V_k \in T$, so that $T$ has a finite sub-cover of $\Omega$, a contradiction.

It is sufficient to consider $n = 2$: if $\mathcal{M}_1$, $\mathcal{M}_2$ are two finite collections of open sets and $B_1, B_2 \in \tau$ are such that $\mathcal{M}_1 \cup \{B_1\}$ and $\mathcal{M}_2 \cup \{B_2\}$ both cover $\Omega$, then in particular, $\mathcal{M}_1 \cup \{B_1\}$ covers $B_2$, so that $\mathcal{M}_1 \cup \{B_1 \cap B_2\}$ covers $B_2$ as well and $\mathcal{M}_1 \cup \{B_1 \cap B_2\} \cup \mathcal{M}_2$ covers $\Omega$.

**3.** Let $\{(\Omega_i, \tau_i)\}_{i\in I}$ is a family of topological spaces. Provide the cartesian product $\prod_{i\in I} \Omega_i$ with the collection $\tau$ of sets consisting of arbitrary unions of sets of the form $\prod_{i\in I} V_i$, where all but a finite number of the $V_i's$ are equal to $\Omega_i$.
Show that $\tau$ is a topology. For some fixed $i \in I$, let $\pi_i : \prod_{i\in I} \Omega_i \to \Omega_i$, $(x_i)_{i\in I} \mapsto x_i \in \Omega_i$ be the **canonical projection**. Show that the collection of sets $\{\pi_i^{-1}(V) \ : \ i \in I \ \text{ and } \ V \in \tau_i\}$ is a sub-basis for $\tau$. Prove that the product topology is then the coarsest topology for which all the maps $\pi_i$ are continuous.

Let $\mathcal{S} := \{S \subset J \ : \ |S| \in \mathbb{J}\}$. Then $\tau$ consists of arbitrary unions of sets in

$$\beta := \{\prod_{i\in S} V_i \prod_{j\in J\setminus S} \Omega_j \ : \ S \in \mathcal{S}, \, V_i \in \tau_i\}.$$

This last collection of sets satisfies

$(B1)$ :

$$\cup_{U\in\beta} = \prod_{i\in J} \Omega_i.$$

Indeed, if $S=\emptyset$, then $\prod_{i\in S} V_i \prod_{j\in J\setminus S} \Omega_j = \prod_{j\in J} \Omega_j$.

$(B2)$ : if $U, U' \in \beta$, then there is an $U'' \in \beta$ so that $U'' \in U' \cap U''$.
Indeed, let $U = \prod_{i\in S} V_i \prod_{j\in J\setminus S} \Omega_j$ and $U' = \prod_{i\in T} V_i' \prod_{j\in J\setminus T} \Omega_j$. Then set $R := S\cup T$ and for $i\in R$ set

$$V_i'' := \begin{cases} V_i \cap V_i' & \text{if } i\in S\cap T, \\ V_i\cap \Omega_i & \text{if } i \in S\setminus T, \\ \Omega_i \cap V_i' & \text{if } i\in T\setminus S. \end{cases}$$

Obviously $R\in\mathcal{S}$ and $V_i''\in\tau_i$ for each $i\in R$. Clearly, $U\cap U' = \prod_{i\in R} V_i'' \prod_{j\in J\setminus R}\Omega_j \in \beta$.

A set $\beta$ satisfying $(B1)$ and $(B2)$ is then a basis for the collection $\tau$ of sets which are arbitrary unions of elements in $\beta$ and $\tau$ is a topology. Indeed,

- $\emptyset, \prod_j \Omega_j \in \tau$:
  For the latter, $\prod_j \Omega_j \in \tau$ by property $(B1)$ and if $S=\{i\}$ for some $i\in J$ and $U_i=\emptyset$, then $\prod_{i\in S} V_i \prod_{j\in J\setminus T}\Omega_j = \emptyset \in \beta \subset \tau$.
- if $\mathcal{F}\subset\tau$, then $\cup_{U\in\mathcal{F}} U \in \tau$.
  This is obvious, since $\cup_{U\in\mathcal{F}} U$ is again an arbitrary union of elements in $\beta$, for any $U\in\mathcal{S}$ is n arbitrary union of elements in $\beta$.
- if $\mathcal{F}\subset\tau$, then $\cap_{U\in\mathcal{F}} U\in\tau$ if $|\mathcal{F}|\in\mathbb{N}$.
  It is obviously enough to prove this if $\mathcal{F}$ consists of only two elements $U, U'$. In this case, let $U = \cup_{B\in\mathcal{C}\subset\beta} B$ and $U' = \cup_{B\in\mathcal{C}'\subset\beta} B$ . Then $U\cap U' = \cup_{B,B'\in\mathcal{C}\cup\mathcal{C}'} B\cap B'$, which by property $(B2)$ is again a union of elements in $\beta$.

$\tau$ is therefore a topology and $\beta$ is a base for $\tau$.

The collection of sets

$$\sigma := \{\pi_i^{-1}(V_i) \; : \; i\in J \text{ and } V_i\in\tau_i\}$$

is a sub-base for $\tau$, since for any $S\in\mathcal{S}$, $\prod_{i\in S} V_i \prod_{j\in J\setminus S}\Omega_j = \cap_{i\in S}\pi_i^{-1}(V_i)$.

If $\tau' \subset \mathcal{P}(\prod_{i\in J}\Omega_i)$ is another topology for which all the maps $\pi_i$ are continuous, then $\tau'$ must by definition contain all sets of the form $\pi_i^{-1}(V_i)$ with $U_i\in\tau_i$. Hence, $\tau'\supset\sigma$.
$\tau'$ being a topology it must contain all finite intersections of elements in $\tau'$, so that it must contain $\beta$.
$\tau'$ being a topology it must contain all arbitrary unions of elements in $\tau'$, so that it must contain $\tau$.
As a consequence, any topology for which all projections $\pi_i$ are continuous contains at least $\tau$. This latter topology is hence the coarsest one for which all projections $\pi_i$ are continuous.

**4.** The **one-point compactification** of $\mathbb{R}^N$ is defined as the set $\dot{\mathbb{R}^N} := \mathbb{R}^N \cup \{*\}$, where $* \notin \mathbb{R}^N$, provided with the topology $\tau$ consisting of the canonical open sets in $\mathbb{R}^N$ together with sets of the form $\{*\} \cup (\mathbb{R}^N \setminus K)$, with $K \subset \mathbb{R}^N$ and $K$ is compact in the canonical topology in $\mathbb{R}^N$.
Prove that $\dot{\mathbb{R}^N}$ is a compact set. Prove that $f \in C(\dot{\mathbb{R}^N})$ iff, when restricted to $\mathbb{R}^N$, $f = \lambda + g$ with $g \in C_0(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$.
Consider $\Omega_L := \prod_{t>0} \dot{\mathbb{R}^N}$, provided with the product topology, and the set of finite functions $C_{\text{fin}}(\Omega_L)$. Prove that $C_{\text{fin}}(\Omega_L) \subset C(\Omega_L)$ and that $C_{\text{fin}}(\Omega_L)$ is uniformly dense in $C(\Omega_L)$.

Let $\mathcal{F} \subset \tau$ be a cover of $\dot{\mathbb{R}^N}$. Then at least one of the sets $U \in \mathcal{F}$ has $*$ as one of its elements. By definition, for such a $U$, we must have $\dot{\mathbb{R}^N} \setminus U = K$ with $K$ a compact set of $\mathbb{R}^N$. But $\cup_{U' \in \mathcal{F}} U' = \dot{\mathbb{R}^N}$, so that $\cup_{U' \in \mathcal{F}\setminus\{U\}} U' \supset K$, which is obviously equivalent to $\cup_{U' \in \mathcal{F}\setminus\{U\}} (\mathbb{R}^N \cap U') \supset K$. $K$ being compact, this is already the case for some finite subset $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G} \cup \{U\}$, a finite set of open sets in $\tau$, covers therefore $\dot{\mathbb{R}^N}$. This shows the compactness of $\dot{\mathbb{R}^N}$.

For a function $f : \dot{\mathbb{R}^N} \to \mathbb{C}$, let $U \subset \mathbb{C}$ be some open set with $\lambda := f(*) \in U$.
If $f$ is continuous, then $f^{-1}(U) \in \tau$ for any such open set $U$. Obviously, $* \in f^{-1}(U)$ and $\dot{\mathbb{R}^N} \setminus f^{-1}(U)$ is compact in $\mathbb{R}^N$.
Let $\epsilon > 0$ and set $B(\lambda, \epsilon) := \{z \in \mathbb{C} : |z - \lambda| < \epsilon\}$. Then $\dot{\mathbb{R}^N} \setminus f^{-1}(B(\lambda, \epsilon))$ is a compact set $K_\epsilon \subset \mathbb{R}^N$ and if $x \notin K_\epsilon$, then $|f(*) - f(x)| < \epsilon$. This is the exact definition of $\lim_{|x|\to\infty} f(x) = \lambda$, or equivalently $\lim_{|x|\to\infty} (f - \lambda)(x) = 0$. Hence, when restricted to $\mathbb{R}^N$, $f = \lambda + g$ with $g := f - \lambda \in C_0(\mathbb{R}^N)$.
Conversely, if $f = \lambda + g$ and $\lambda \in \mathbb{C}$, $g \in C_0(\mathbb{R}^N)$, then by definition, $\lim_{|x|\to\infty} g(x) = \lim_{|x|\to\infty} (f - \lambda)(x) = 0$. By definition, this means that for any given $\epsilon > 0$, there is a compact set $K_\epsilon \subset \mathbb{R}^N$, so that $x \notin K_\epsilon$ implies $|g(x)| = |f(x) - \lambda| < \epsilon$. Therefore, for $B(\lambda, \epsilon) := \{z \in \mathbb{C} : |z - \lambda| < \epsilon\}$, one has $f^{-1}(B(\lambda, \epsilon) = \mathbb{R}^N \setminus K_\epsilon$.
If $U \subset \mathbb{R}^N$ is open and if $\lambda \notin U$, then there is some $\epsilon > 0$ so that $B(\lambda, \epsilon) \cap U = \emptyset$. Hence, $f^{-1}(U) \cap f^{-1}(B(\lambda, \epsilon)) = \emptyset$, so that $f^{-1}(U) \subset K_\epsilon$.
If one now extends $g + \lambda$ to $\dot{\mathbb{R}^N}$ by imposing $f(*) = \lambda$, then $f$ is certainly continuous on $\dot{\mathbb{R}^N}$ for the topology $\tau$.

If $F \in C_{\text{fin}}(\Omega_L)$, then by definition there is some finite set $S \subset \mathbb{R}_+$ and a continuous function $f : (\dot{\mathbb{R}^N})^{|S|} \to \mathbb{C}$, so that

$$\Omega_L \ni (w_t)_{t\geq 0} \mapsto F((w_t)_{t\geq 0}) := f((w_s)_{s\in S}).$$

If $U \subset \mathbb{C}$ is open, then $F^{-1}(U) = f^{-1}(U) \times \prod_{t\in\mathbb{R}_+\setminus S} \dot{\mathbb{R}}^N$ and it remains to be shown, that $f^{-1}(U)$ is open in $(\dot{\mathbb{R}^N})^{|S|}$ for the product topology. Since $f$ is continuous on $(\dot{\mathbb{R}^N})^{|S|}$ then by definition, $f^{-1}(U)$ is open for the product topology on $(\dot{\mathbb{R}^N})^{|S|}$, so that $F^{-1}(U) = f^{-1}(U) \times \prod_{t\in\mathbb{R}_+\setminus S} \dot{\mathbb{R}}^N$ is open for the product topology in $\Omega_L$ and $F$ is continuous. Therefore, $C_{\text{fin}}(\Omega_L) \subset C(\Omega_L)$.

To prove the uniform density of $C_{\text{fin}}(\Omega_L)$ in $C(\Omega_L)$ we shall verify the conditions of Stone & Weierstrass's theorem:

- $C_{\text{fin}}(\Omega_L)$ is an algebra over $\mathbb{C}$, closed under complex conjugation.

  Let $R, S, T \subset \mathbb{R}_+$ be finite sets and let $F, G, H \in C_{\text{fin}}(\Omega_L)$ be defined by

$$\Omega_L \ni (w_t)_{t\geq 0} \mapsto F((w_t)_{t\geq 0}) := f((w_t)_{t\in R}),$$
$$\Omega_L \ni (w_t)_{t\geq 0} \mapsto G((w_t)_{t\geq 0}) := g((w_t)_{t\in S}),$$
$$\Omega_L \ni (w_t)_{t\geq 0} \mapsto H((w_t)_{t\geq 0}) := h((w_t)_{t\in T}),$$

  with $f \in C((\dot{\mathbb{R}}^N)^{|R|})$, $g \in C((\dot{\mathbb{R}}^N)^{|S|})$ and $h \in C((\dot{\mathbb{R}}^N)^{|T|})$. Then, $D := R \cup S \cup T$ is a finite subset of $\mathbb{R}_+$ as well and $f \times (g+h)$ is a continuous function on $(\dot{\mathbb{R}}^N)^{|D|}$ with

$$(f \times (g+h))((w_t)_{t\in D}) = f((w_t)_{t\in R}) \times (g((w_t)_{t\in S}) + h((w_t)_{t\in T})).$$

  The lift on $\Omega_L$ is then the finite function

$$(F \times (G+H))((w_t)_{t\geq 0}) = (f \times (g+h))((w_t)_{t\in D}).$$

  The complex conjugate $\overline{F}$ of $F$ is then simply given by

$$\overline{F}((w_t)_{t\geq 0}) := \overline{f}((w_t)_{t\in R}) \quad \Rightarrow \quad \overline{F} \in C_{\text{fin}}(\Omega_L),$$

  since clearly $\overline{f} \in C((\dot{\mathbb{R}}^N)^{|R|})$.
  It is also obvious that $F \equiv 0 \in C_{\text{fin}}(\Omega_L)$, so that $C_{\text{fin}}(\Omega_L)$ is indeed an algebra over $\mathbb{C}$ of continuous functions on $\Omega_L$ that is closed under complex conjugation.

- $C_{\text{fin}}(\Omega_L)$ does not vanish on $\Omega_L$.

  This is clear since the constant function $1((w_t)_{t\geq 0}) = 1$ is obviously an element of $C_{\text{fin}}(\Omega_L)$ and does vanish nowhere.

- $C_{\text{fin}}(\Omega_L)$ separates points on $\Omega_L$.

  If $(w_t)_{t\geq 0} \neq (v_t)_{t\geq 0}$, then $\exists s \in \mathbb{R}_+$ so that $w_s \neq u_s$. Since $C(\dot{\mathbb{R}}^N)$ manifestly separates the points in $\dot{\mathbb{R}}^N$, there is a function $f \in C(\dot{\mathbb{R}}^N)$ with $f(w_s) \neq f(u_s)$. Set then $F \in C_{\text{fin}}(\Omega_L)$ as $F((w_t)_{t\geq 0}) := f(w_s)$, which is by construction a finite function, separating $(w_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$.

We may hence apply the theorem by Stone & Weierstrass on the compact set $\Omega_L$ and conclude, that $C_{\text{fin}}(\Omega_L)$ is dense in $C(\Omega_L)$ for the supremum norm $\| \ \|_\infty$.

**5.** For a fixed $x_0 \in \mathbb{R}^N$ and for some $F_{f,t_1,\ldots,t_m} \in C_{\text{fin}}(\Omega_L)$, we define

$$I_{x_0}(F_{f,t_1,\ldots,t_m}) := \int_{\mathbb{R}^{mN}} p^{x_0}_{t_1,\ldots,t_m}(x_1,\ldots,x_m) f(x_1,\ldots,x_m)\mu_L(dx_1\times\ldots\times dx_m),$$

$$p^{x_0}_{t_1,\ldots,t_m}(x_1,\ldots,x_m) = \prod_{k=1}^m \frac{1}{(2\pi(t_k-t_{k-1}))^{N/2}} e^{-\frac{(x_k-x_{k-1})^2}{2(t_k-t_{k-1})}}, \quad t_0 = 0.$$

Check that $I_{x_0}(1) := 1$. For $F_{f,t_1,\ldots,t_m}$ and $0 < t$ with $\{t_1,\ldots,t_m\}\cap\{t\} = \emptyset$, if $G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}(w) := F_{f,t_1,\ldots,t_m}(w)$, check that

$$I_{x_0}(G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}) = I_{x_0}(F_{f,t_1,\ldots,t_m}).$$

For some constant $C > 0$ and for $\epsilon, t > 0$, prove that

$$\mathbb{E}_{x_0,W}(\mathbb{1}_{\{w\in\Omega_L\,:\,|w_t-x_0|>\epsilon\}}) \le C\frac{\sqrt{t}}{\epsilon}e^{-\frac{\epsilon^2}{2Nt}}.$$

If we set

$$\rho(\epsilon,\delta) := \sup\{\mu_{x_0,W}(\{w\in\Omega_L\,:\,|w_t-x_0|>\epsilon\})\,:\,0<t\le\delta\},$$

$$\text{then verify that}\quad \lim_{\delta\to 0^+}\frac{1}{\delta}\rho(\epsilon,\delta) = 0.$$

The constant function 1 on $\Omega_L$ is described by some $F_{f,t_1,\ldots,t_m}$ with $f$ the constant unit function on $(\mathbb{R}^N)^m$, $\{t_1,\ldots,t_m\}\subset\mathbb{R}_+$ and $t_0 = 0$. But then

$$\begin{aligned}
I_{x_0}(1) &= \int_{\mathbb{R}^{mN}} p^{x_0}_{t_1,\ldots,t_m}(x_1,\ldots,x_m) f(x_1,\ldots,x_m)\mu_L(dx_1\times\ldots\times dx_m)\\
&= \int_{\mathbb{R}^{mN}} p^{x_0}_{t_1,\ldots,t_m}(x_1,\ldots,x_m)\mu_L(dx_1\times\ldots\times dx_m)\\
&= \int_{\mathbb{R}^{mN}} \prod_{k=1}^m \frac{1}{(2\pi(t_k-t_{k-1}))^{N/2}} e^{-\frac{(x_k-x_{k-1})^2}{2(t_k-t_{k-1})}}\mu_L(dx_1\times\ldots\times dx_m).
\end{aligned}$$

The last integral is equal to its Riemann integral and successive integration over all variables yield integrations of the type $\int_{\mathbb{R}^N}\frac{1}{(2\pi(t-s))^{N/2}}e^{-\frac{(x-y)^2}{2(t-s)}}d^Nx$, which are all normalised gaussians, so that $I_{x_0}(1) = 1$.

For $F_{f,t_1,\ldots,t_m}$ and $0 < t$ with $\{t_1,\ldots,t_m\}\cap\{t\} = \emptyset$, if $G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}(w) := F_{f,t_1,\ldots,t_m}(w)$, then $G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m} = G_{g,t_1,\ldots,t_k,t,t_{k+1},t_m}$ with $g \in C((\mathbb{R}^N)^{m+1})$ and

$g(x_1,\ldots,x_k,x,x_{k+1},\ldots,x_m)=f(x_1,\ldots,x_m)$. Note also, that

$$p^{x_0}_{t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}(x_1,\ldots,x_k,x,x_{k+1},\ldots,x_m)$$
$$=\left(\prod_{l=1}^{k}\frac{1}{(2\pi(t_l-t_{l-1}))^{N/2}}e^{-\frac{(x_l-x_{l-1})^2}{2(t_l-t_{l-1})}}\right)\times\frac{1}{(2\pi(t-t_k))^{N/2}}e^{-\frac{(x-x_k)^2}{2(t-t_k)}}$$
$$\times\frac{1}{(2\pi(t_{k+1}-t))^{N/2}}e^{-\frac{(x_{k+1}-x)^2}{2(t_{k+1}-t)}}\times\left(\prod_{l=k+2}^{m}\frac{1}{(2\pi(t_l-t_{l-1}))^{N/2}}e^{-\frac{(x_l-x_{l-1})^2}{2(t_l-t_{l-1})}}\right).$$

Integrating this probability measure along the variable $x$ yields the convolution of two gaussian probability distributions. This convolution may be computed by Fourier transforms or by a direct computation involving completion of squares:

$$\int_{\mathbb{R}^N}\frac{1}{(2\pi(t-t_k))^{N/2}}e^{-\frac{(x-x_k)^2}{2(t-t_k)}}\frac{1}{(2\pi(t_{k+1}-t))^{N/2}}e^{-\frac{(x_{k+1}-x)^2}{2(t_{k+1}-t)}}\mu_L(dx)$$
$$=\frac{1}{(2\pi)^N}\frac{1}{(t-t_k)^{N/2}}\frac{1}{(t_{k+1}-t)^{N/2}}\int_{\mathbb{R}^N}e^{-\frac{1}{2}\left(\frac{(x-x_k)^2}{(t-t_k)}+\frac{(x_{k+1}-x)^2}{(t_{k+1}-t)}\right)}\mu_L(dx)$$
$$\underset{y=x-x_k}{=}\frac{1}{(2\pi)^N}\frac{1}{\big((t-t_k)(t_{k+1}-t)\big)^{N/2}}\int_{\mathbb{R}^N}e^{-\frac{1}{2}\left(\frac{y^2}{(t-t_k)}+\frac{((x_{k+1}-x_k)-y)^2}{(t_{k+1}-t)}\right)}\mu_L(dy)$$
$$=\frac{1}{(2\pi)^N}\frac{1}{\big((t-t_k)(t_{k+1}-t)\big)^{N/2}}\int_{\mathbb{R}^N}e^{-\frac{1}{2(t-t_k)(t_{k+1}-t)}\left((t_{k+1}-t)y^2+(t-t_k)((x_{k+1}-x_k)-y)^2\right)}\mu_L(dy)$$
$$=\frac{1}{(2\pi)^N}\frac{e^{-\frac{(x_{k+1}-x_k)^2}{2(t_{k+1}-t)}}}{\big((t-t_k)(t_{k+1}-t)\big)^{N/2}}\int_{\mathbb{R}^N}e^{-\frac{1}{2(t-t_k)(t_{k+1}-t)}\left((t_{k+1}-t_k)y^2-2(t-t_k)(x_{k+1}-x_k)y\right)}\mu_L(dy)$$
$$=\frac{1}{(2\pi)^N}\frac{e^{-\frac{(x_{k+1}-x_k)^2}{2(t_{k+1}-t)}}e^{\frac{(t-t_k)(x_{k+1}-x_k)^2}{2(t_{k+1}-t_k)(t_{k+1}-t)}}}{\big((t-t_k)(t_{k+1}-t)\big)^{N/2}}\int_{\mathbb{R}^N}e^{-\frac{(t_{k+1}-t_k)}{2(t-t_k)(t_{k+1}-t)}\left(y-\frac{(t-t_k)}{(t_{k+1}-t_k)}(x_{k+1}-x_k)\right)^2}\mu_L(dy)$$
$$=\frac{e^{-\frac{(x_{k+1}-x_k)^2}{2(t_{k+1}-t_k)}}}{\big(2\pi(t_{k+1}-t_k)\big)^{N/2}}.$$

Consequently,

$$\int_{\mathbb{R}^N}p^{x_0}_{t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}(x_1,\ldots,x_k,x,x_{k+1},\ldots,x_m)\mu_L(dx)$$
$$=p^{x_0}_{t_1,\ldots,t_m}(x_1,\ldots,x_m)$$

and

$$I_{x_0}(G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m})$$
$$=\int_{\mathbb{R}^{mN}}p^{x_0}_{t_1,\ldots,t_k,t,t_{k+1},t_m}(x_1,\ldots,x_m)g(x_1,\ldots,x_k,x,x_{k+1},\ldots,x_m)\mu_L(dx_1\times\ldots dx\times\ldots dx_m)$$
$$=\int_{\mathbb{R}^{mN}}p^{x_0}_{t_1,\ldots,t_k,t,t_{k+1},t_m}(x_1,\ldots,x_m)f(x_1,\ldots,x_k,x_{k+1},\ldots,x_m)\mu_L(dx_1\times\ldots\times dx_m)\mu_L(dx)$$
$$=\int_{\mathbb{R}^{mN}}p^{x_0}_{t_1,\ldots,t_k,t_{k+1},t_m}(x_1,\ldots,x_m)f(x_1,\ldots,x_m)\mu_L(dx_1\times\ldots\times dx_m)$$
$$=I_{x_0}(F_{f,t_1,\ldots,t_m}).$$

If $N=1$, one has for $f(y)=\mathbb{1}_{|y-x_0|>\epsilon}$

$$\mathbb{E}_{x_0,W}(\mathbb{1}_{\{w\in\Omega_L\,:\,|w_t-x_0|>\epsilon\}}) = I_{x_0}(F_{f,t})$$
$$= \int_{\mathbb{R}} p_t^{x_0}(x)\mathbb{1}_{|x-x_0|>\epsilon}\mu_L(dx) = \int_{\mathbb{R}\setminus[x_0-\epsilon,x_0+\epsilon]} \frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-x_0)^2}{2t}}\mu_L(dx)$$
$$= \int_{\mathbb{R}\setminus[-\epsilon,+\epsilon]} \frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}\mu_L(dy) = \frac{2}{\sqrt{2\pi t}}\int_\epsilon^\infty e^{-\frac{y^2}{2t}}\mu_L(dy)$$
$$\leq \frac{2}{\sqrt{2\pi t}}\int_\epsilon^\infty \frac{y}{\epsilon}e^{-\frac{y^2}{2t}}\mu_L(dy) = \sqrt{\frac{2t}{\pi}}\frac{1}{\epsilon}e^{-\frac{\epsilon^2}{2t}}.$$

For a general dimension $N\in\mathbb{N}$, note that $|x-x_0|>\epsilon$ with $x,x_0\in\mathbb{R}^N$ implies that $|x_k-x_{0,k}|>\frac{\epsilon}{\sqrt{N}}$ for at least one component $k=1,\ldots,N$. Therefore, if $|w_t-x_0|>\epsilon$, then $w_t\in\cup_{k=1}^N\{w_t\in\mathbb{R}^N\,:\,|w_{t,k}-x_{0,k}|>\frac{\epsilon}{\sqrt{N}}\}$. Each of these sets in this finite union has a Wiener measure dominated by $\sqrt{2tN/\pi}\frac{1}{\epsilon}e^{-\frac{\epsilon^2}{2Nt}}$, so that

$$\rho(\epsilon,\delta)\leq\sqrt{\frac{2N^3}{\pi}}\frac{\sqrt{\delta}}{\epsilon}e^{-\frac{\epsilon^2}{2N\delta}}.$$

Substituting $A$ for $1/\delta$, one gets

$$0\leq\lim_{\delta\to0}\frac{1}{\delta}\rho(\epsilon,\delta)\leq C\lim_{\delta\to0}\frac{1}{\sqrt{\delta}\epsilon}e^{-\frac{\epsilon^2}{2N\delta}}$$
$$= C\lim_{A\to\infty}\frac{\sqrt{A}}{\epsilon}e^{-\frac{\epsilon^2A}{2N}}=0.$$

**6.** Let $\delta,\epsilon>0$ and consider a finite set $S\subset\mathbb{R}_+$ so that $\forall t\in S$, $|t-t_{\min(S)}|\leq\delta$. For $t\in S$, let

$$C_{t,\epsilon,S}:=\{w\in\Omega_L\,:\,|w_t-w_{\max(S)}|>\frac{\epsilon}{2}\},$$
$$D_{t,\epsilon,S}:=\{w\in\Omega_L\,:\,|w_t-w_{\min(S)}|>\epsilon\text{ and }\forall s\in S\text{ with }s<t,\,|w_s-w_{\min(S)}|\leq\epsilon\}.$$

Shoe then that $\forall t\in S$,

$$\mu_{x_0,W}(C_{t,\epsilon,S}\cap D_{t,\epsilon,S})\leq\rho(\frac{\epsilon}{2},\delta)\mu_{x_0,W}(D_{t,\epsilon,S}).$$

Let $f(x,w):=\mathbb{1}_{|x-w|>\frac{\epsilon}{2}}$, so that $\mathbb{1}_{C_{t,\epsilon,S}}=F_{f,t,\max(S)}$.
Similarly, let $\{t_1,\ldots,t_m\}=S\cap]\min(S),t[$ and define

$$g(y,x_1,\ldots,x_m,x):=\mathbb{1}_{|x-y|>\epsilon}\prod_{k=1}^m\mathbb{1}_{|x_k-y|\leq\epsilon}.$$

Then $\mathbb{1}_{D_{t,\epsilon,S}}=G_{g,\min(S),t_1,\ldots,t_m,t}$ and

$$\mu_{x_0,W}(C_{t,\epsilon,S}\cap D_{t,\epsilon,S})$$
$$=\int_{\mathbb{R}^{N(m+3)}}p^{x_0}_{\min(S),t_1,\ldots,t_m,t,\max(S)}(y,x_1,\ldots,x_m,x,w)g(y,x_1,\ldots,x_m,x)f(x,w)\mu_L(dy\times\ldots\times dw).$$

Observe now that

$$p^{x_0}_{\min(S),t_1,\ldots,t_m,t,\max(S)}(y,x_1,\ldots,x_m,x,w) = p^{x_0}_{\min(S),t_1,\ldots,t_m,t}(y,x_1,\ldots,x_m,x)p^{x}_{t,\max(S)}(x,w).$$

We now may calculate the integral over $w$ first and notice, that the result is bounded by $\rho(\frac{\epsilon}{2},\delta)$. Integration over the remaining variables result in $\mu_{x_0,W}(D_{t,\epsilon,S})$ which yields the announced inequality.

**7.** Let $\delta,\epsilon > 0$ and consider a finite set $S \subset \mathbb{R}_+$ so that $\forall t \in S$, $|t - t_{\min(S)}| \leq \delta$. Let

$$A_{\epsilon,S} := \{w \in \Omega_L \,:\, \exists s \in S \text{ s.t. } |w_s - w_{\min(S)}| > \epsilon\}.$$

Prove then that

$$\mu_{x_0,W}(A_{\epsilon,S}) \leq 2\rho(\frac{\epsilon}{2},\delta).$$

Define

$$B_{\epsilon,S} := \{w \in \Omega_L \,.:\, |w_{\min(S)} - w_{\max(S)}| > \frac{\epsilon}{2}\},$$

$$\text{for } t \in S\ , \quad C_{t,\epsilon,S} := \{w \in \Omega_L \,:\, |w_t - w_{\max(S)}| > \frac{\epsilon}{2}\},$$

$$D_{t,\epsilon,S} := \{w \in \Omega_L \,:\, |w_t - w_{\min(S)}| > \epsilon \text{ and } \forall s \in S \text{ with } s < t,\ |w_s - w_{\min(S)}| \leq \epsilon\}.$$

If $w \in A_{\epsilon,S}$, then $w \in D_{t,\epsilon,S}$ for some $t \in S$.
If $w \notin B_{\epsilon,S}$ and if for some $t \in S$ $w \in D_{t,\epsilon,S}$, then $w \in C_{t,\epsilon,S}$, since $w$ has to move a distance at least $\frac{\epsilon}{2}$ to go back from outside the ball of radius $\epsilon$ into the ball of radius $\frac{\epsilon}{2}$. Therefore

$$A_{\epsilon,S} \subset B_{\epsilon,S} \bigcup \left( \bigcup_{t\in S} C_{t,\epsilon,S} \cap D_{t,\epsilon,S} \right).$$

Thus,

$$\begin{aligned}\mu_{x_0,W}(A_{\epsilon,S}) &\leq \mu_{x_0,W}(B_{\epsilon,S}) + \sum_{t\in S} \mu_{x_0,W}(C_{t,\epsilon,S} \cap D_{t,\epsilon,S}) \\ &\leq \mu_{x_0,W}(B_{\epsilon,S}) + \rho(\frac{\epsilon}{2},\delta) \sum_{t\in S} \mu_{x_0,W}(D_{t,\epsilon,S})\end{aligned}$$

by the previous exercice. Since $D_{t,\epsilon,S}$ and $D_{t',\epsilon,S}$ are by definition disjoint for $S \ni t \neq t' \in S$, one has

$$\mu_{x_0,W}(A_{\epsilon,S}) \leq \mu_{x_0,W}(B_{\epsilon,S}) + \rho(\frac{\epsilon}{2},\delta) \leq 2\rho(\frac{\epsilon}{2},\delta).$$

**8.** Let $\delta,\epsilon > 0$ and consider $0 < t_0 < t_1$ with $t_1 - t_0 \leq \delta$. Define

$$E_{t_0,t_1,\epsilon} := \{w \in \Omega_L \,:\, \exists s,t \in [t_0,t_1] \text{ s.t. } |w_s - w_t| > 2\epsilon\}.$$

Prove then that

$$\mu_{x_0,W}(E_{t_0,t_1,\epsilon}) \leq 2\rho(\frac{\epsilon}{2},\delta).$$

Consider some finite set $S \in [t_0, t_1]$ with $t_0, t_1 \in S$ and notice, that if one defines

$$E_{\epsilon,S} := \{w \in \Omega_L \ : \ \exists t, s \in S \ \text{ s.t. } \ |w_s - w_t| > 2\epsilon\},$$

then $E_{\epsilon,S} \subset A_{\epsilon,S}$ with

$$A_{\epsilon,S} := \{w \in \Omega_L \ : \ \exists s \in S \ \text{ s.t. } \ |w_s - w_{\min(S)}| > \epsilon\},$$

since if $|w_s - w_t| > 2\epsilon$, then $|w_s - w_{t_0}|, |w_t - w_{t_0}| \leq \epsilon$ cannot both hold. Hence, by the previous exercice,

$$\mu_{x_0,W}(E_{\epsilon,S}) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

We now are going to make use of the regularity of the measure $\mu_{x_0,W}$ and note that the aforementioned sets $E_{\epsilon,S}$ are open in the product topology for any finite set $S \subset [t_0, t_1]$ with $t_0, t_1 \in S$.
If we consider the collection of open sets

$$\Gamma_{\epsilon,t_0,t_1} := \{E_{\epsilon,S} \ : \ S \subset [t_0, t_1] \ \text{ is a finite set with } t_0, t_1 \in S\},$$

then

$$E_{t_0,t_1,\epsilon} = \bigcup_{V \in \Gamma_{\epsilon,t_0,t_1}} V$$

and by the regularity of the measure $\mu_{x_0,W}$,

$$\mu_{x_0,W}(E_{t_0,t_1,\epsilon}) = \sup\{\mu_{x_0,W}(E_{\epsilon,S})\} \leq 2\rho(\frac{\epsilon}{2}, \delta).$$