Physics of Life

PHYS-468

Fourier Transform

Henning Stahlberg,
LBEM, IPHYS, SB, EPFL

Jean Baptiste Joseph Fourier

France, 1768 - 1830

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x} \, dx$$

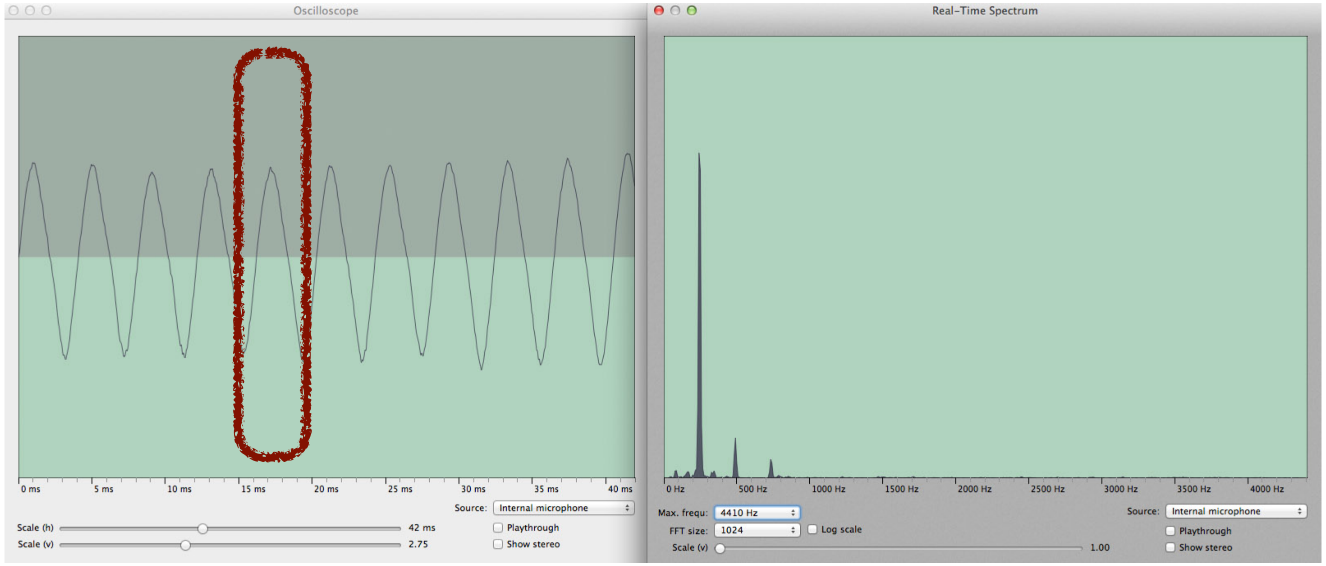
$$F(u) = \frac{1}{2\pi} \int f(x) \cdot \left[\cos(2\pi \cdot u \cdot x) - i \cdot \sin(2\pi \cdot u \cdot x) \right] dx$$

$$F(u) = FT\left(f(x) \right)$$

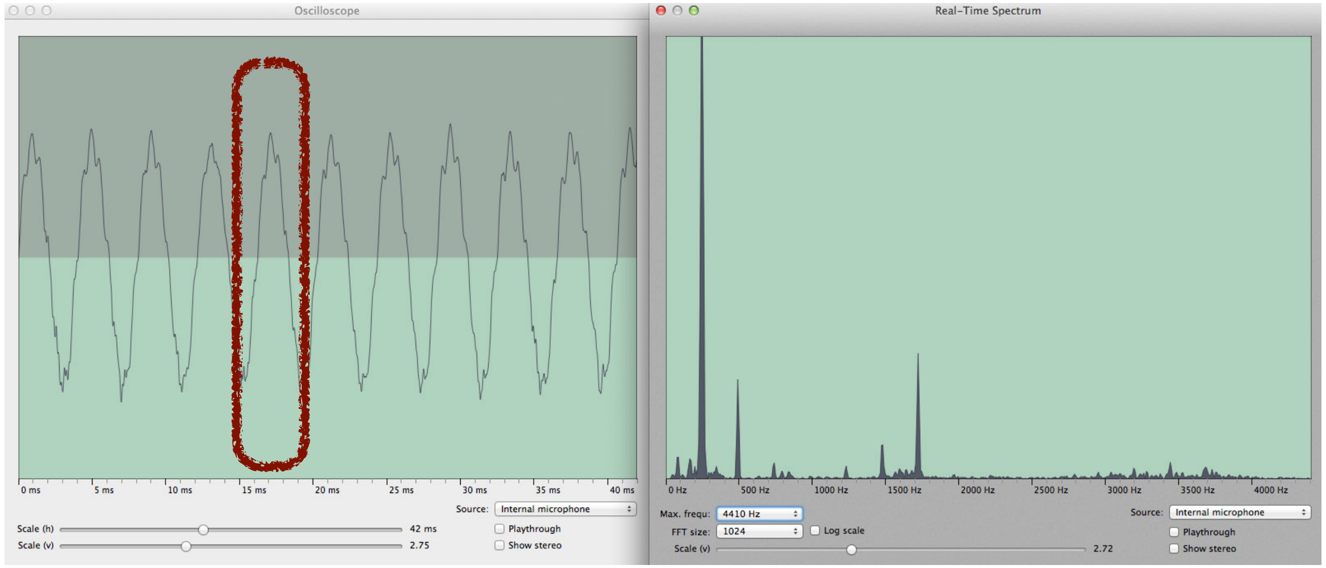
$$PS(u) = \left| F(u) \right|^2$$

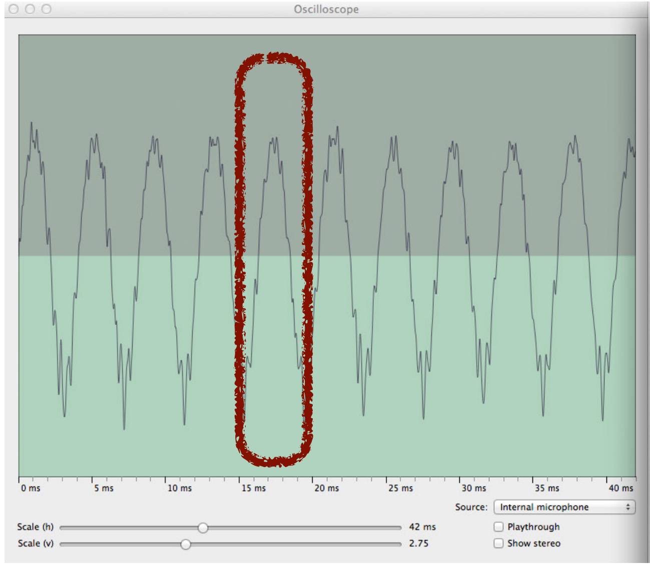
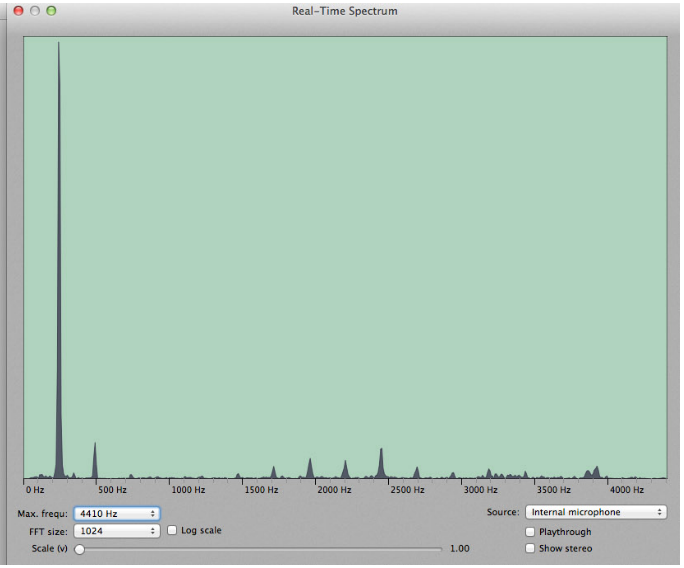
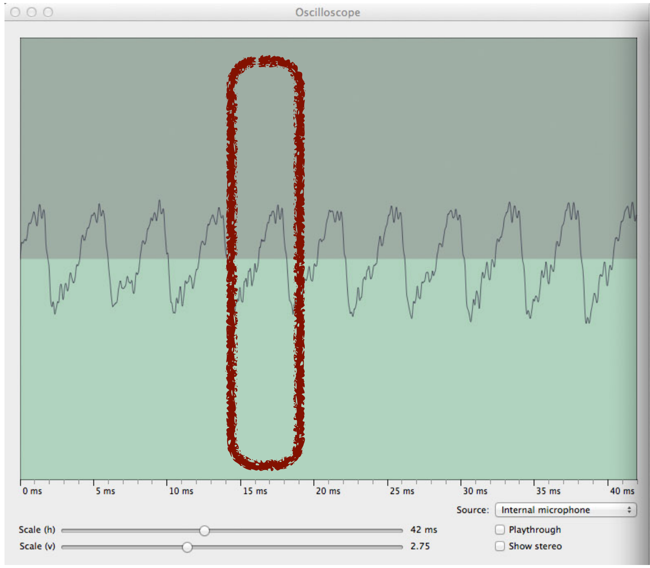
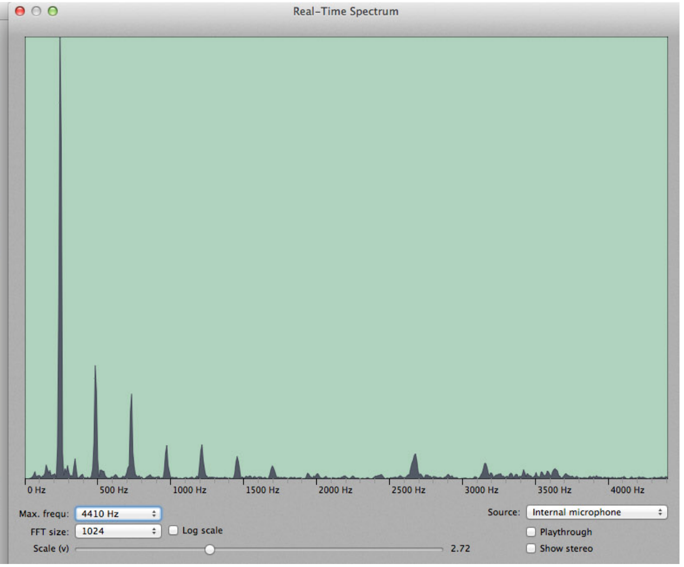
Real Space | **Fourier Space**

Almost Sinus Wave



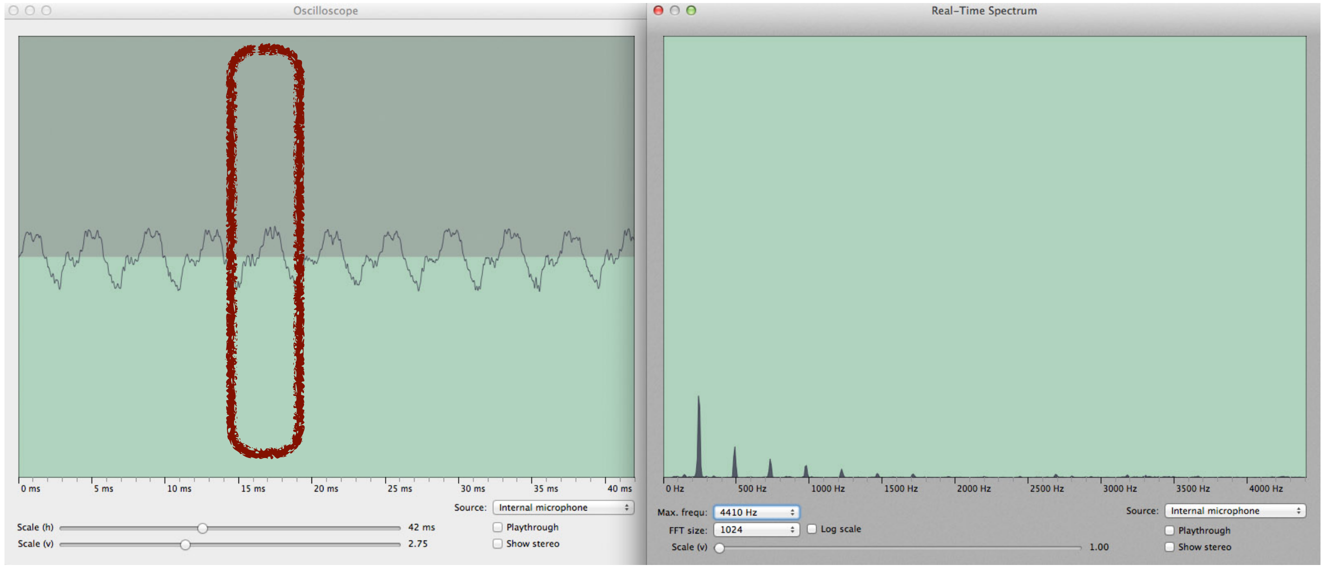
Some Higher Orders



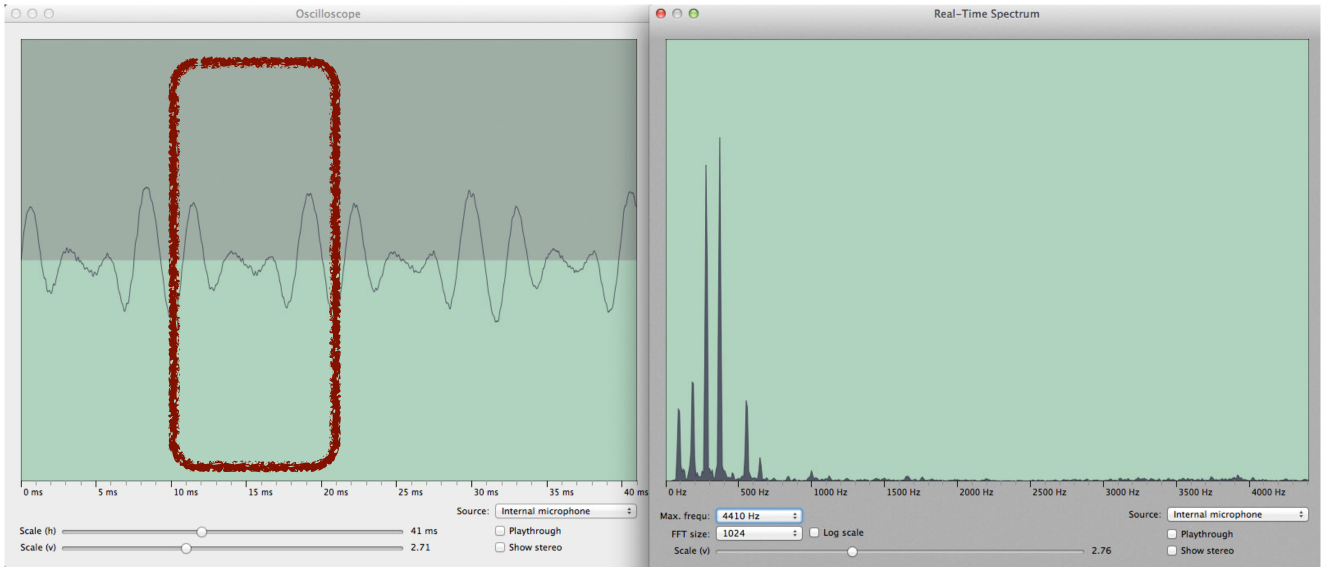
	Real Space	Fourier Space
More Higher Orders		
Low Contrast (small Amplitude) and Higher orders		

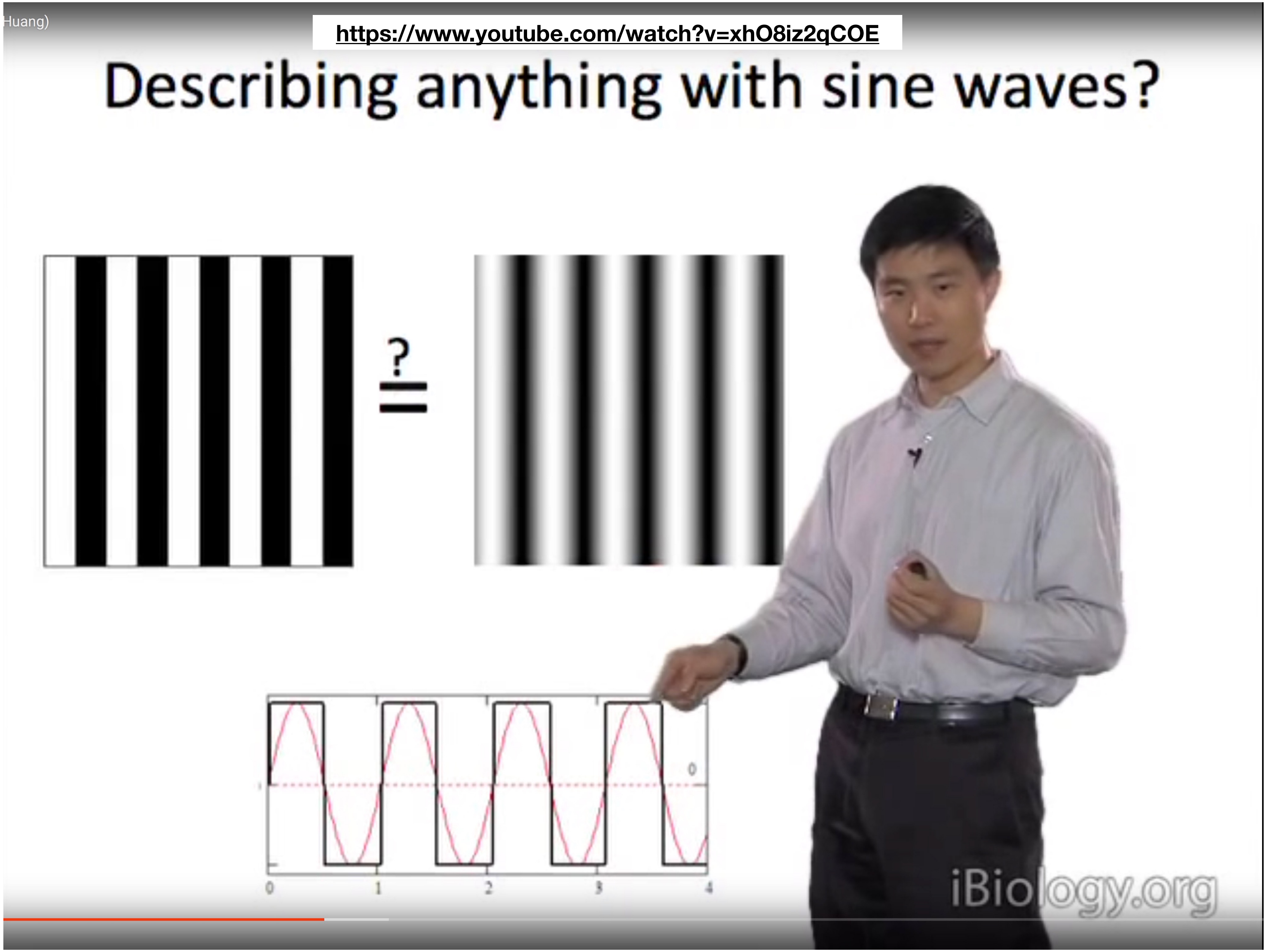
Real Space | Fourier Space

Low Amplitudeand higher orders

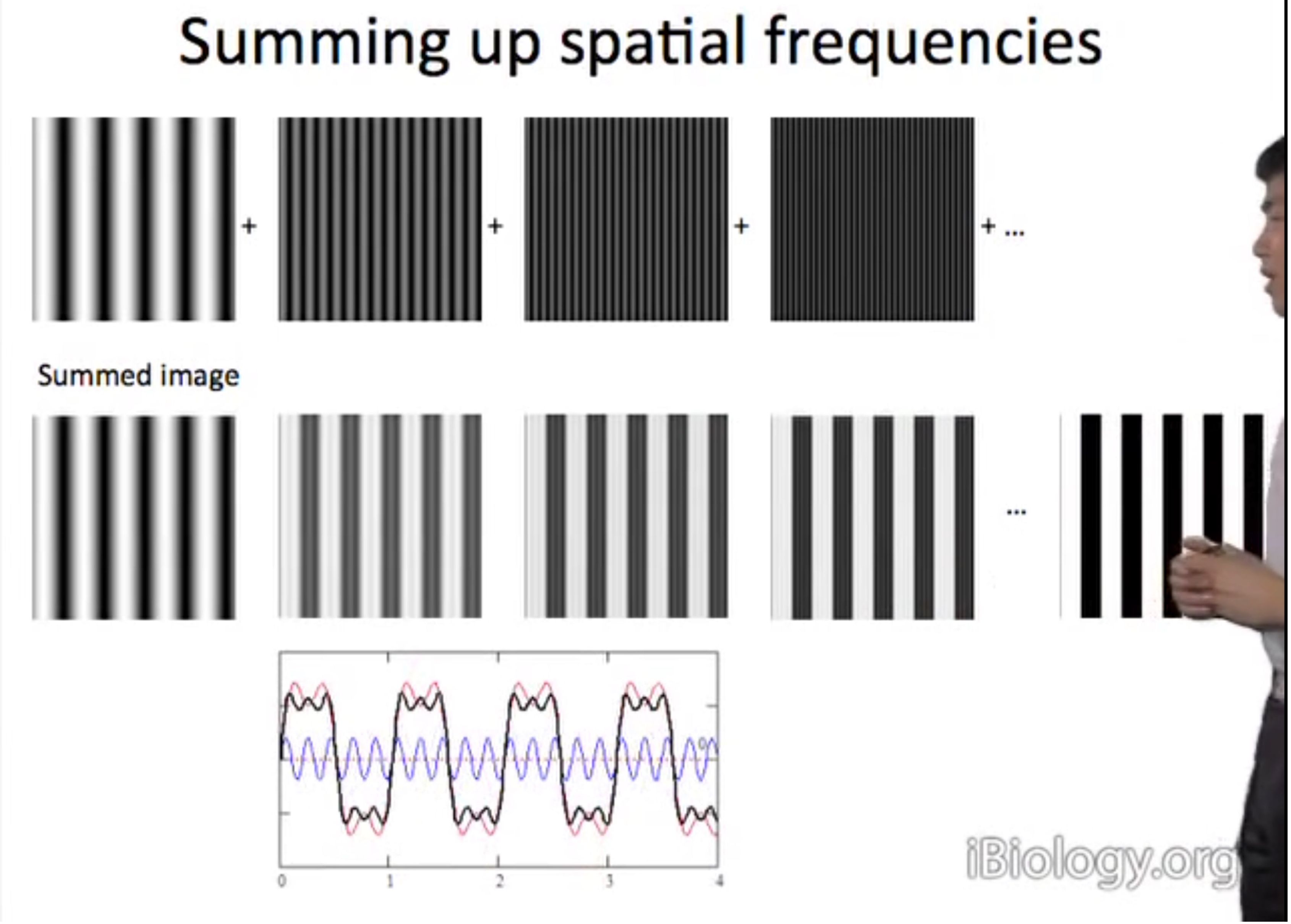


Large structure, or lower base frequency, and higher orders

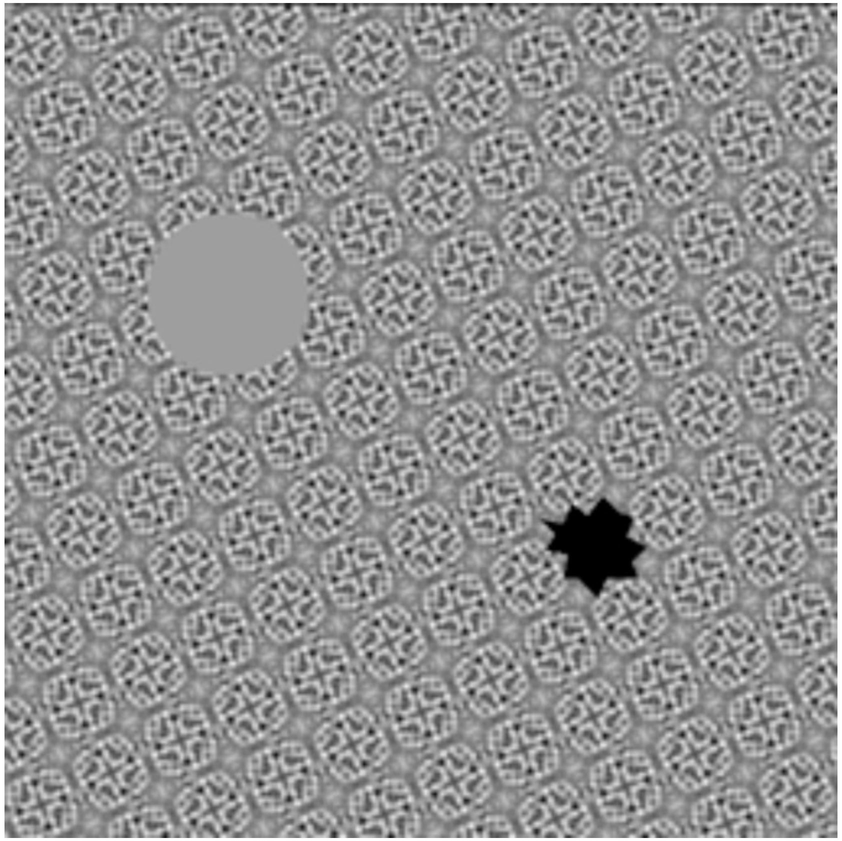



Huang)
https://www.youtube.com/watch?v=xhO8iz2qCOE
Describing anything with sine waves?
?
=
0
1
2
3
4
iBiology.org

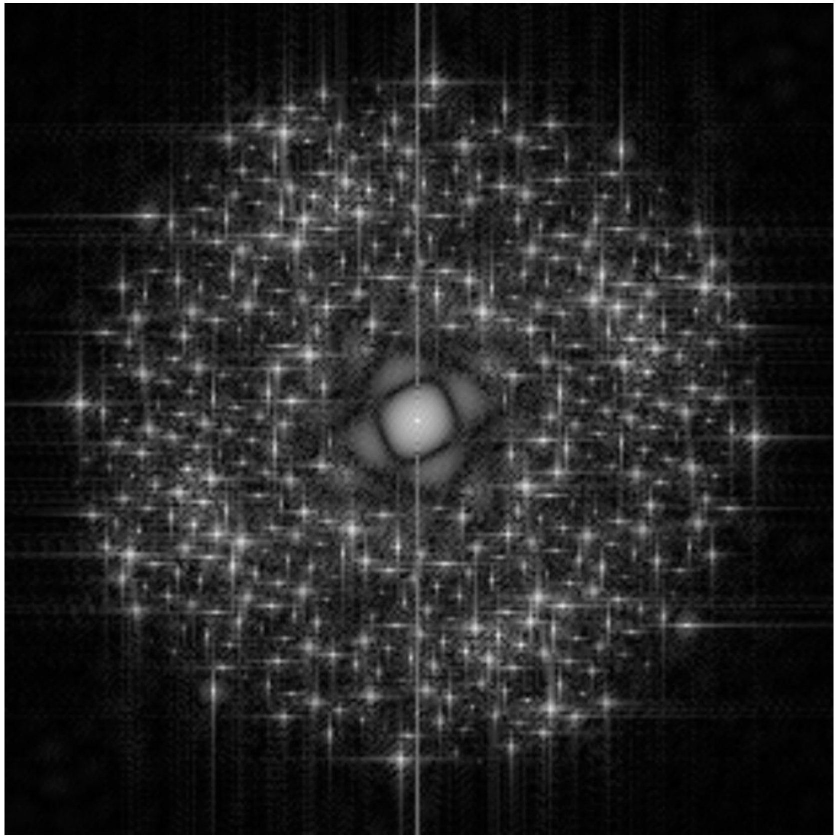
https://www.youtube.com/watch?v=xhO8iz2qCOE



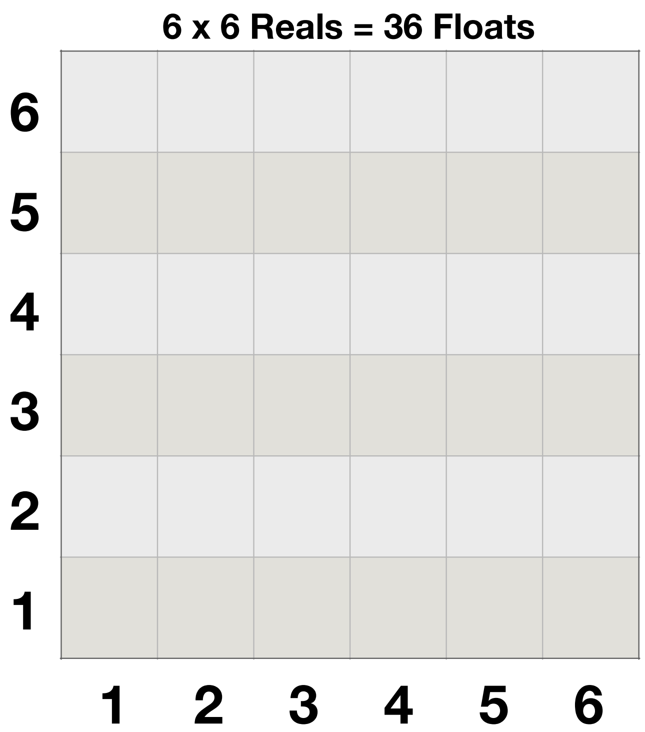
Real Space

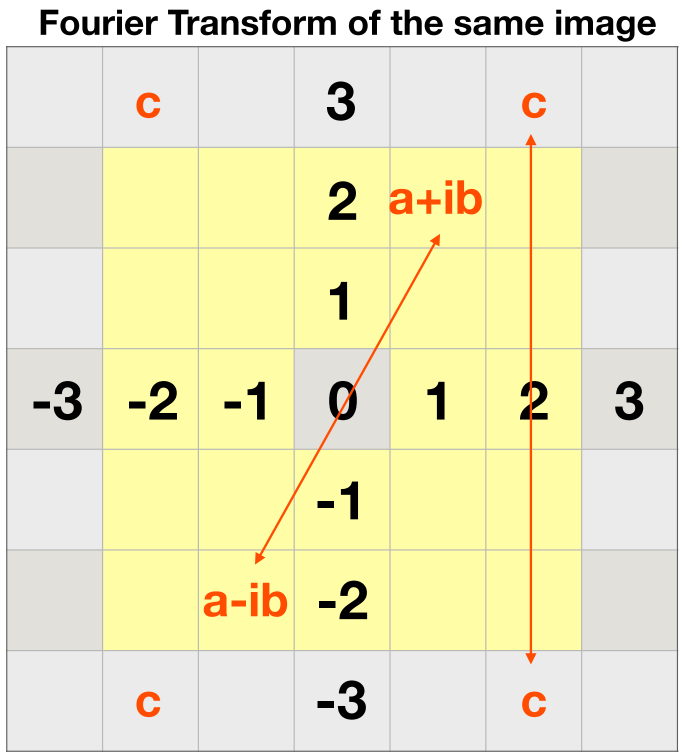


Fourier Transform



2D Fourier Transforms in the Computer Memory

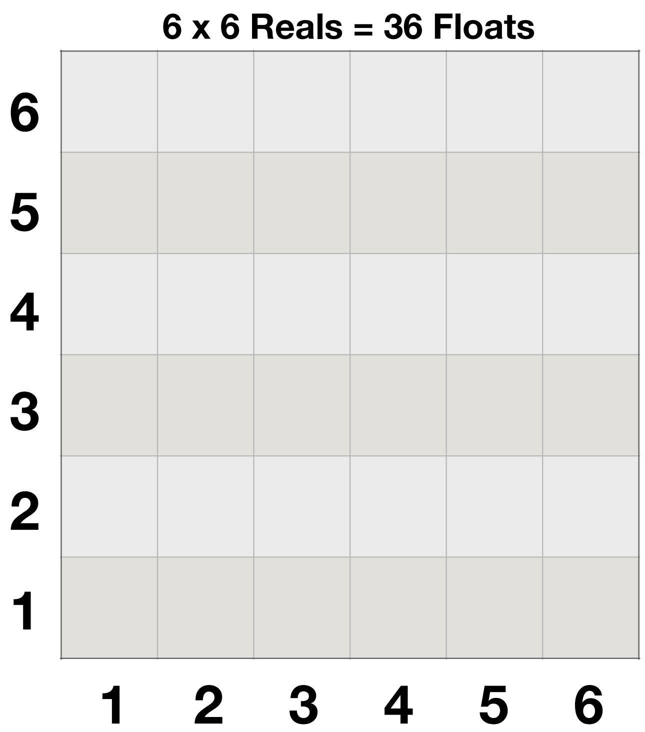


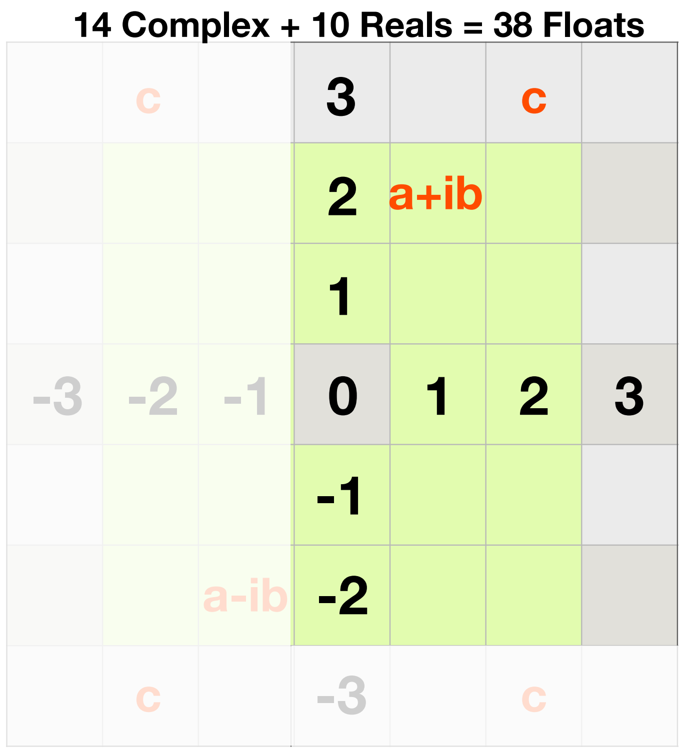


- The FT of real functions (e.g. images) are Hermitian: for every point (a+bi), there is a corresponding point (a-bi)
- For an N × N pixel image, the Fourier transform has the dimensions N/2+1 × N
- The positive Nyquist and negative Nyquist values are the same, and they are real.

(after John Rubinstein, NRAMM 2014)

2D Fourier Transforms in the Computer Memory





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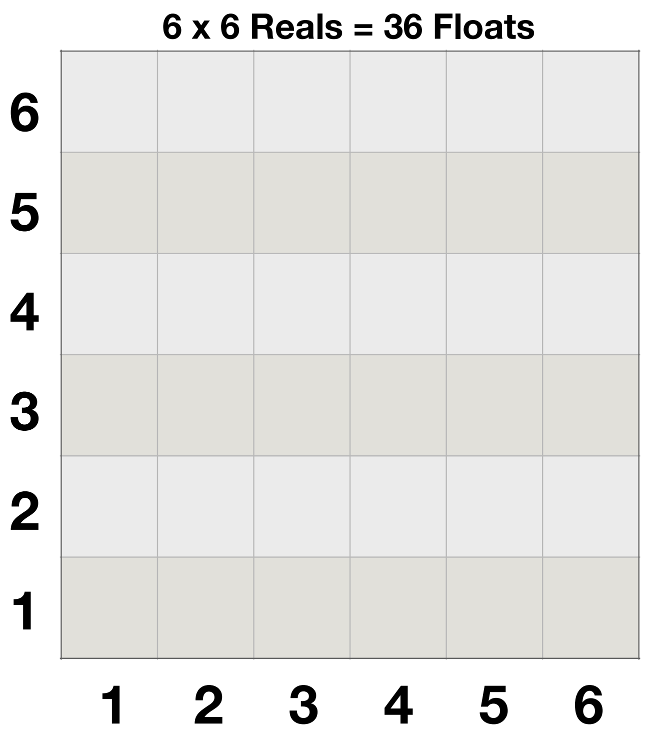

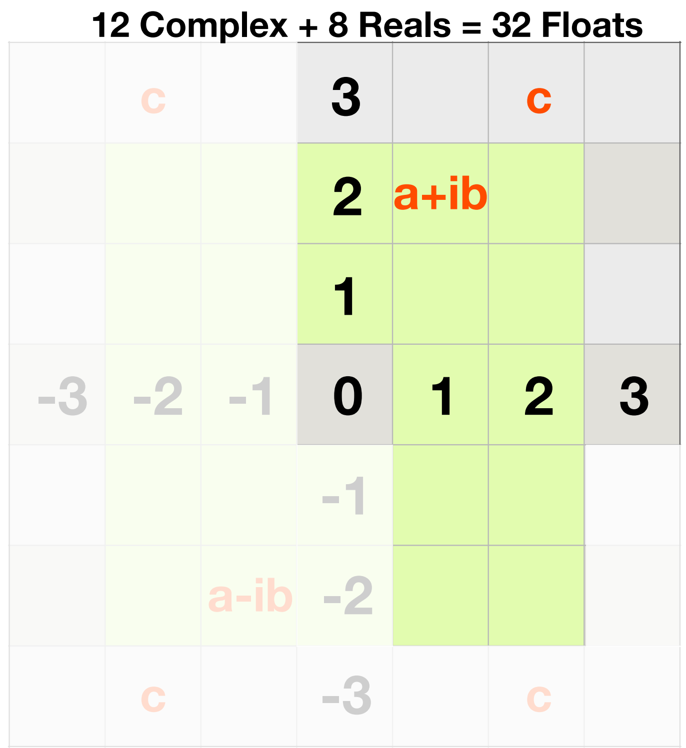

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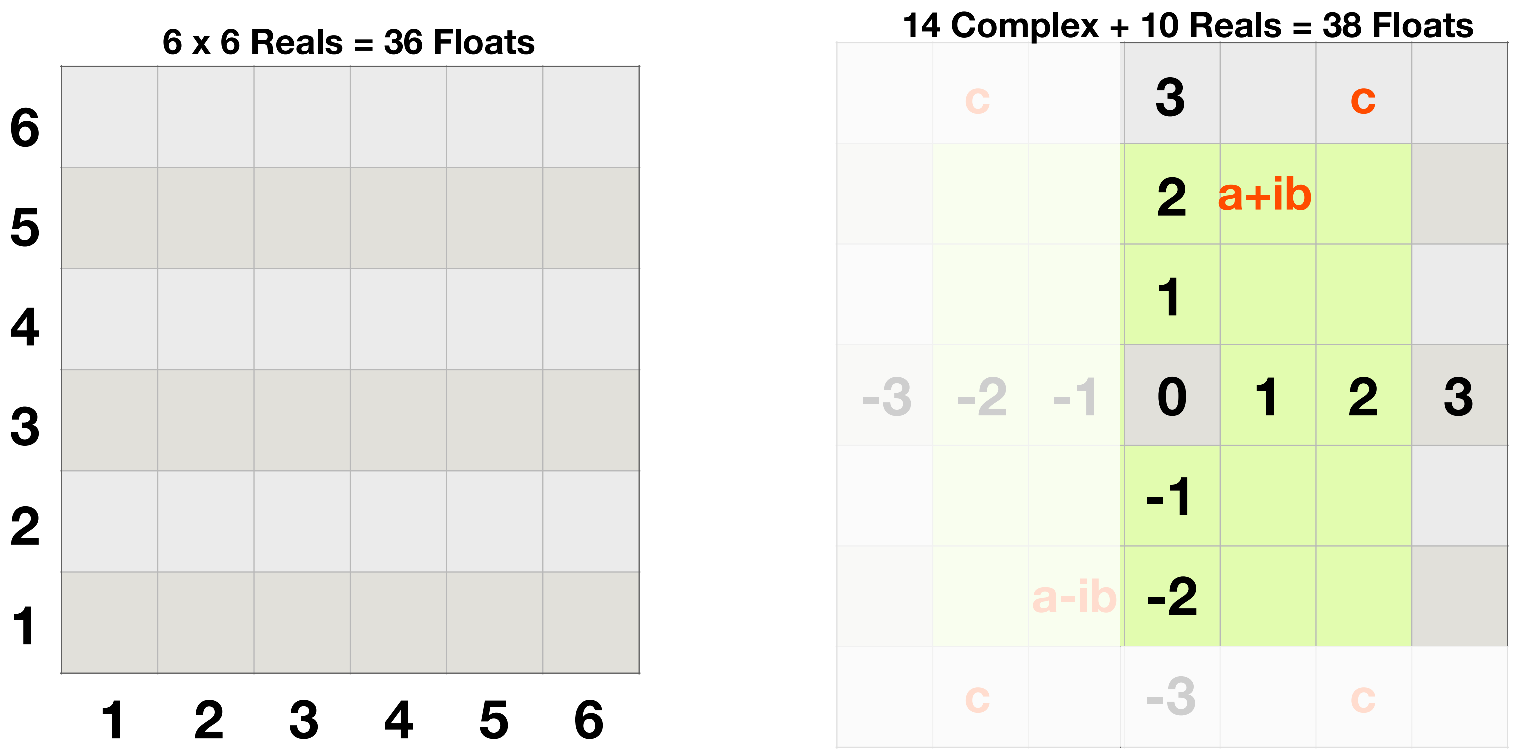

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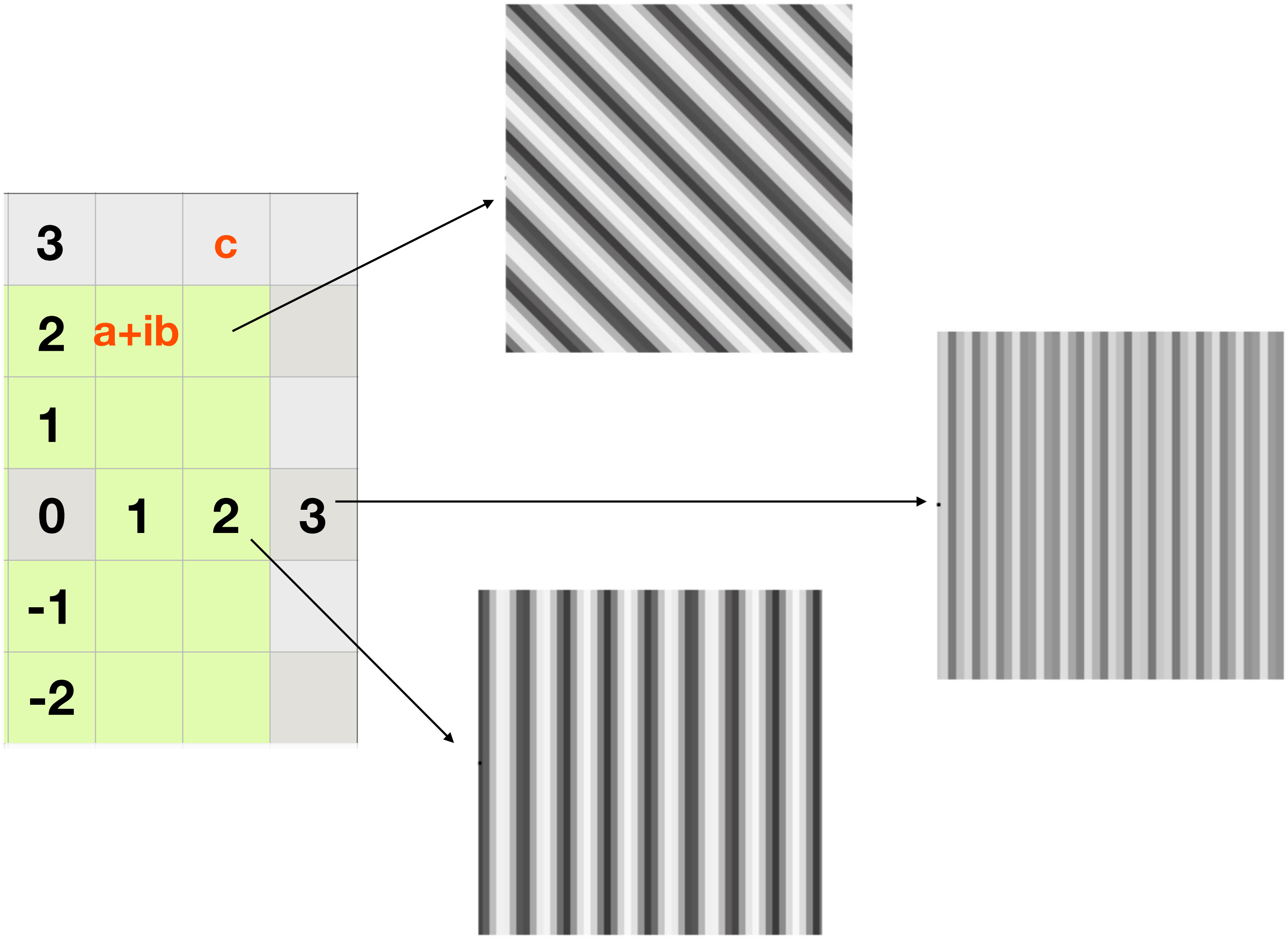
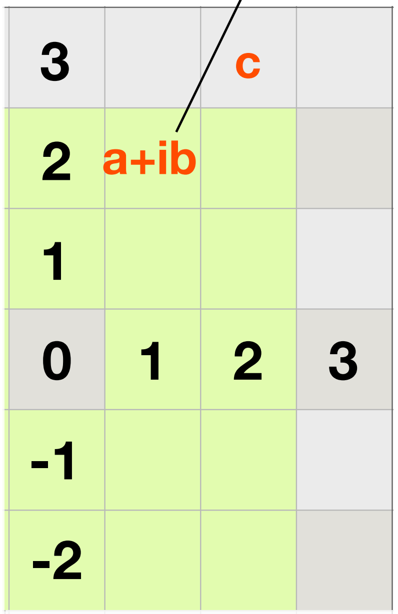
2D Fourier Transforms in the Computer Memory

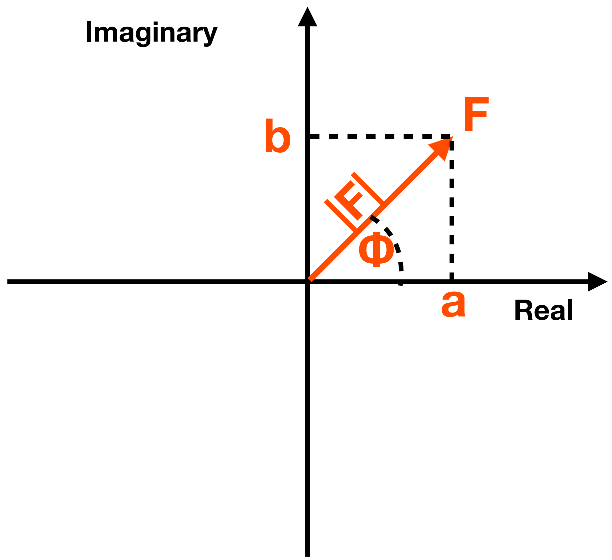


(after John Rubinstein, NRAMM 2014)

2D Fourier Transforms in the Computer Memory

a+ib = “Cosinus-Term (Real)” and “Sinus-Term (Imaginary)”





Argand Diagram:
Polar Coordinate Transform:

$F = |a+ib|$
$\Phi = \text{atan}\ (b/a)$

(F,Φ) = “Amplitude” and “Phase”

(after John Rubinstein, NRAMM 2014)

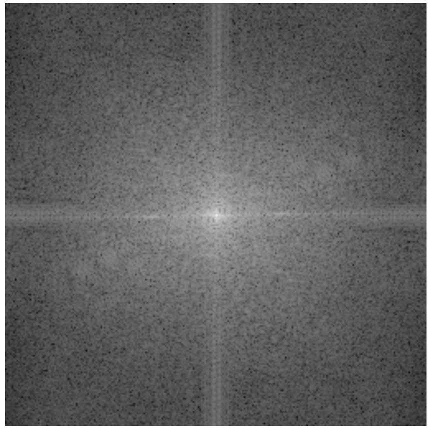
Real-space image <=> Fourier Transform

Real space

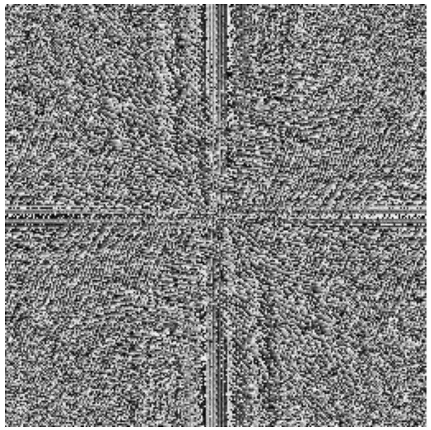
FT

FT^{-1}

Fourier space



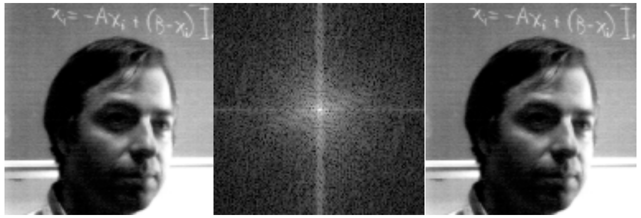
Amplitudes



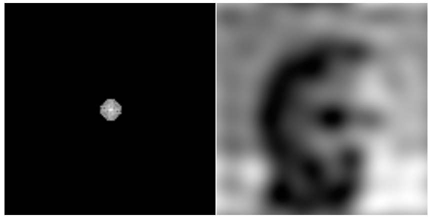
Phases

Basics of image processing: Fourier filters of images

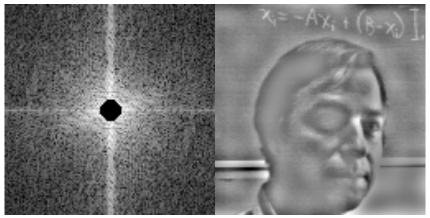
Real -->Fourier -->Real



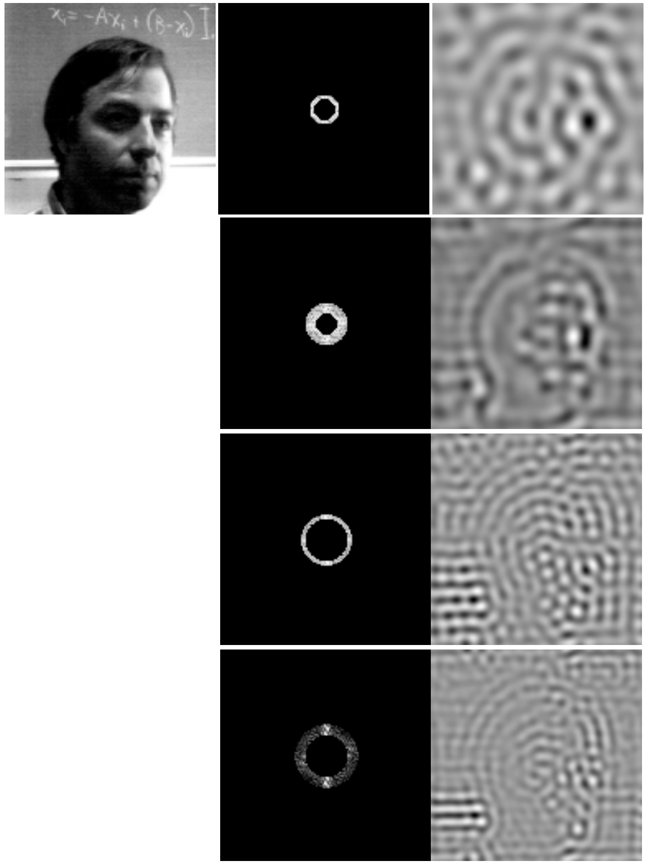
Low-pass filter



High-pass filter

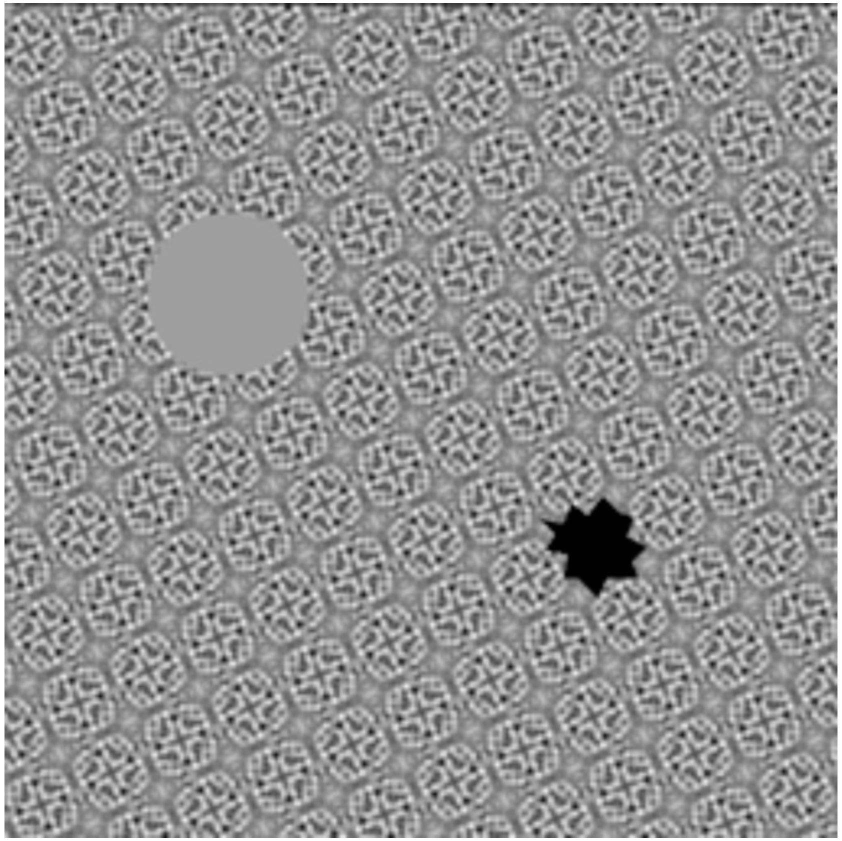


Band-pass

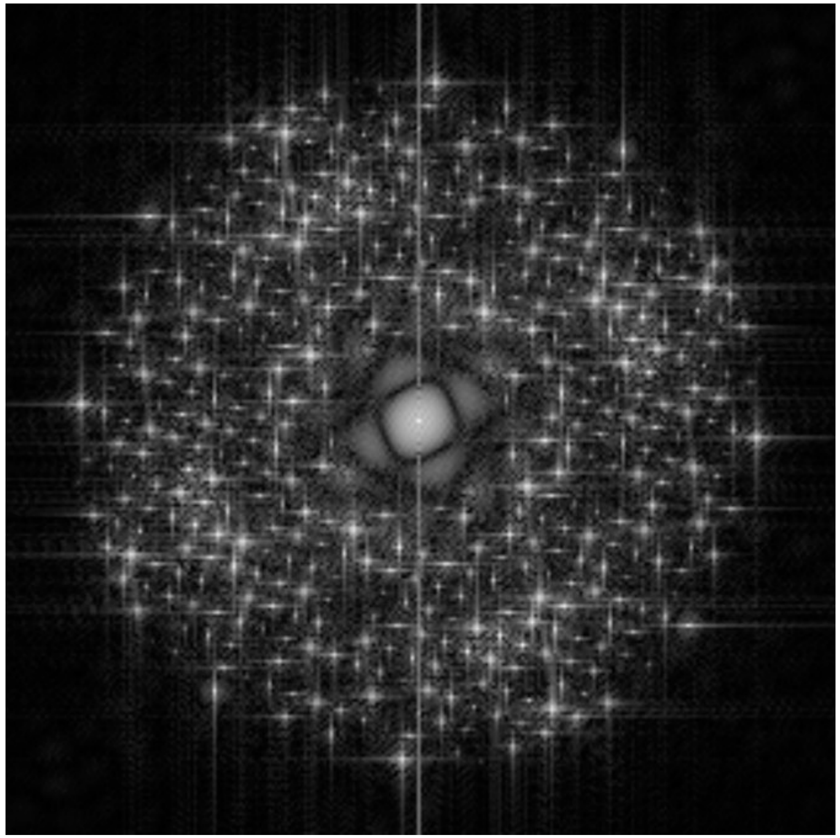


http://sharp.bu.edu/~slehar/fourier/fourier.html

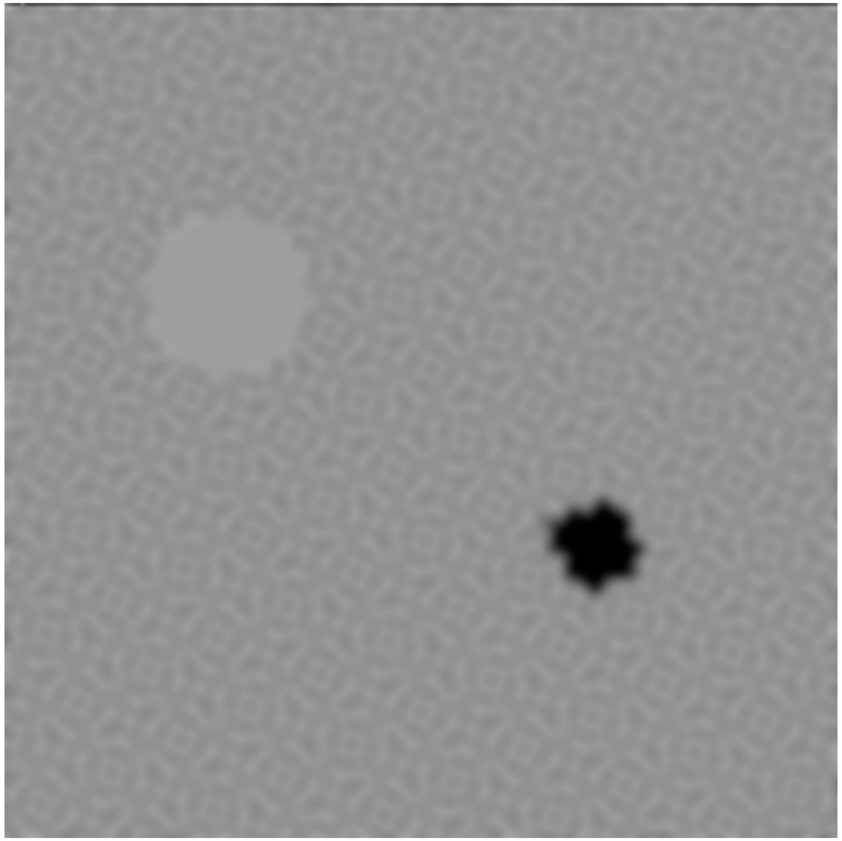
Real Space



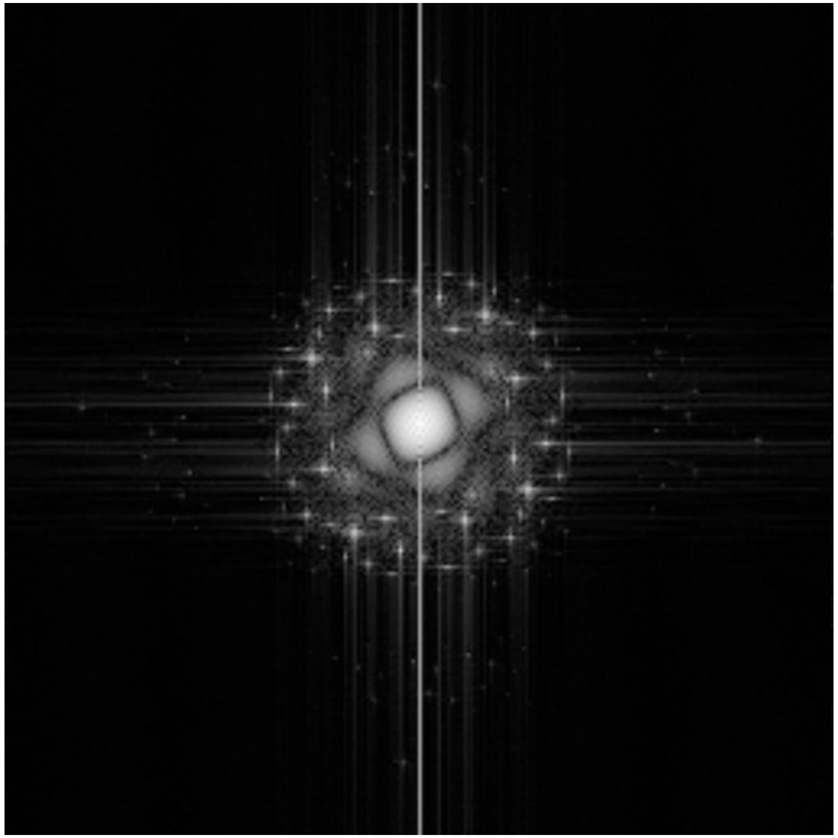
Fourier Transform



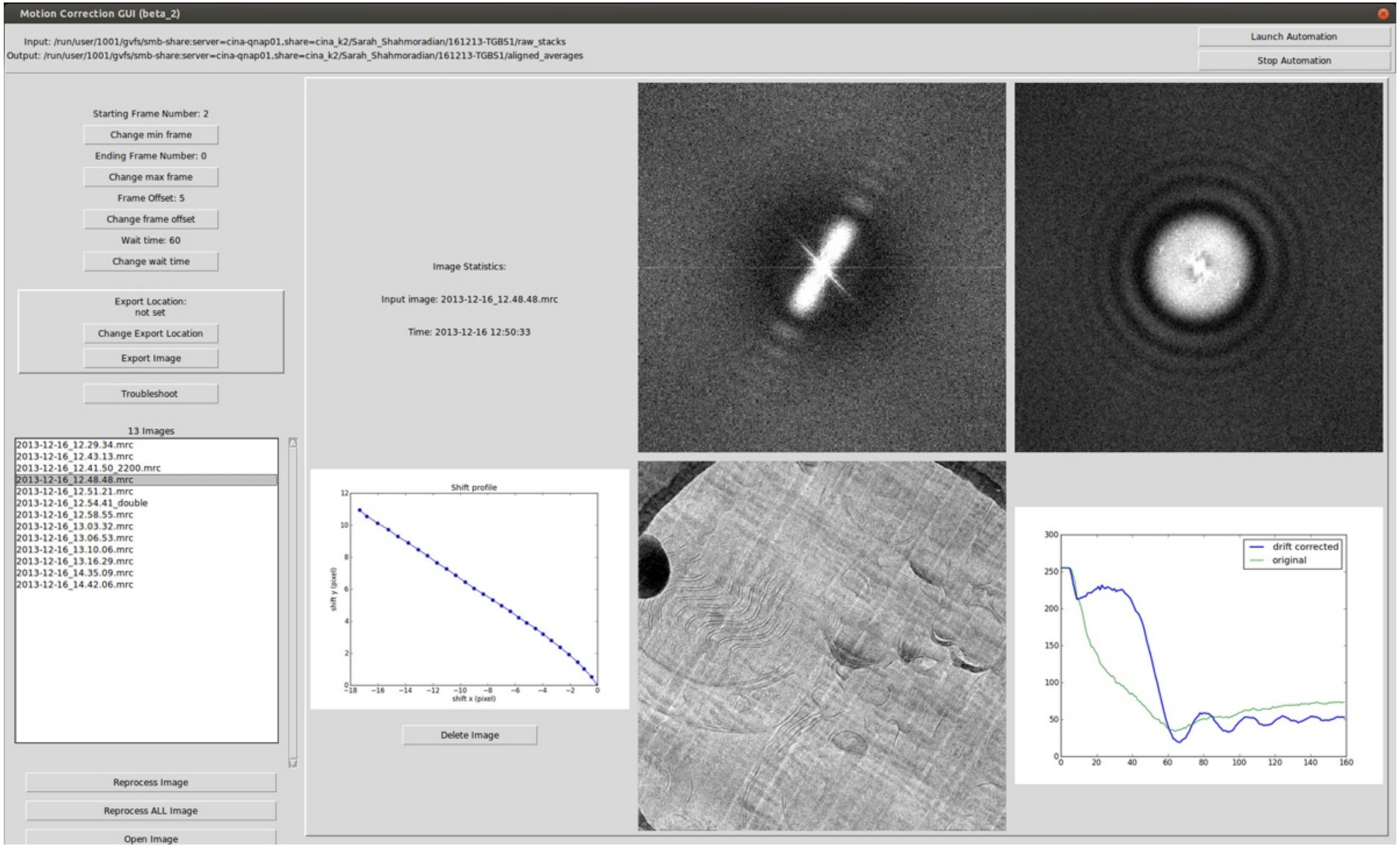
Real Space



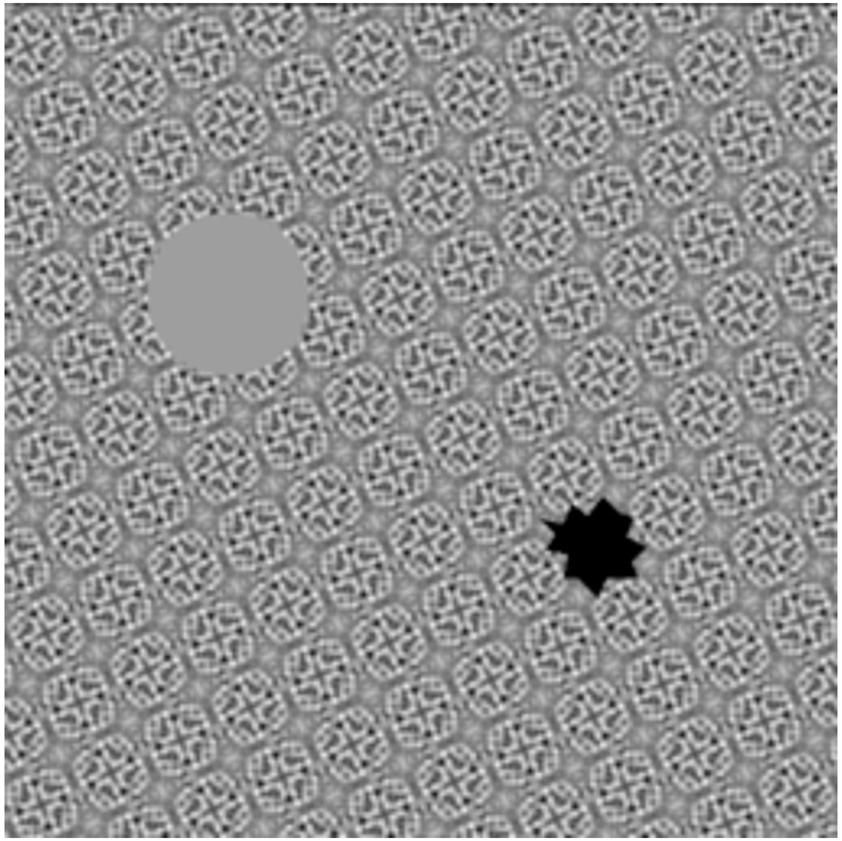
Fourier Transform



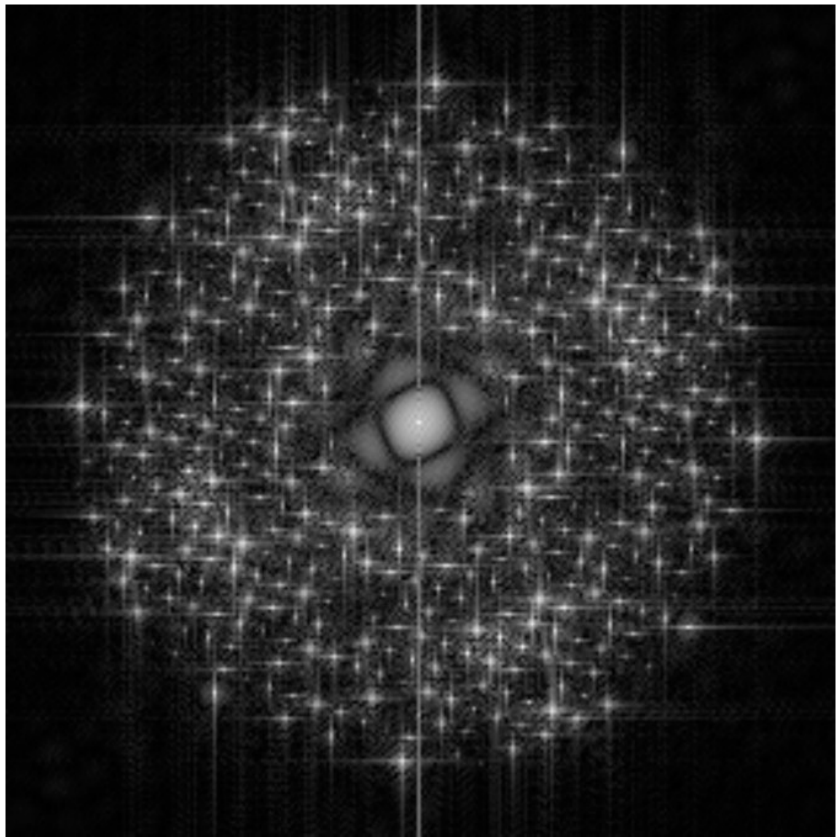
2dx_automator (2013):
Automated movie frame alignment and averaging, based on the MotionCorr(1) (Cheng lab) alignment tool.



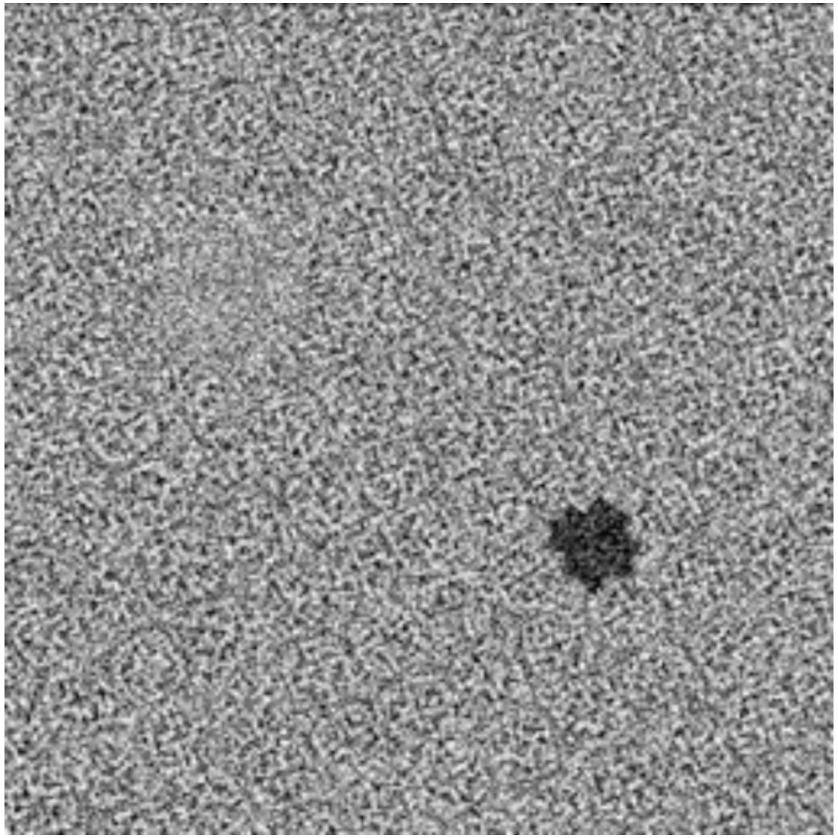
Real Space



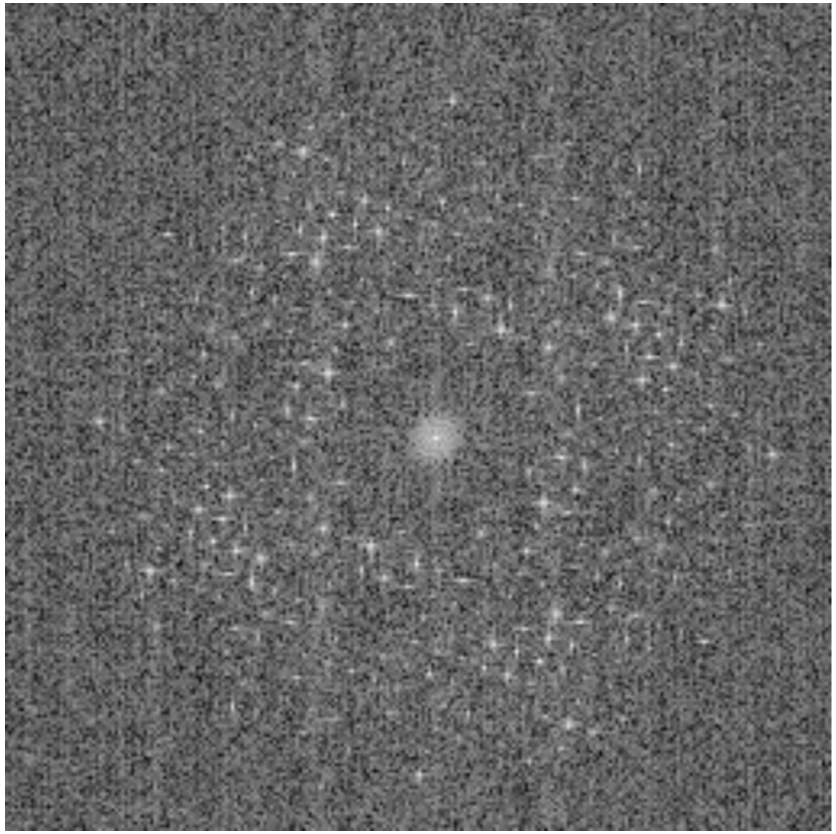
Fourier Transform



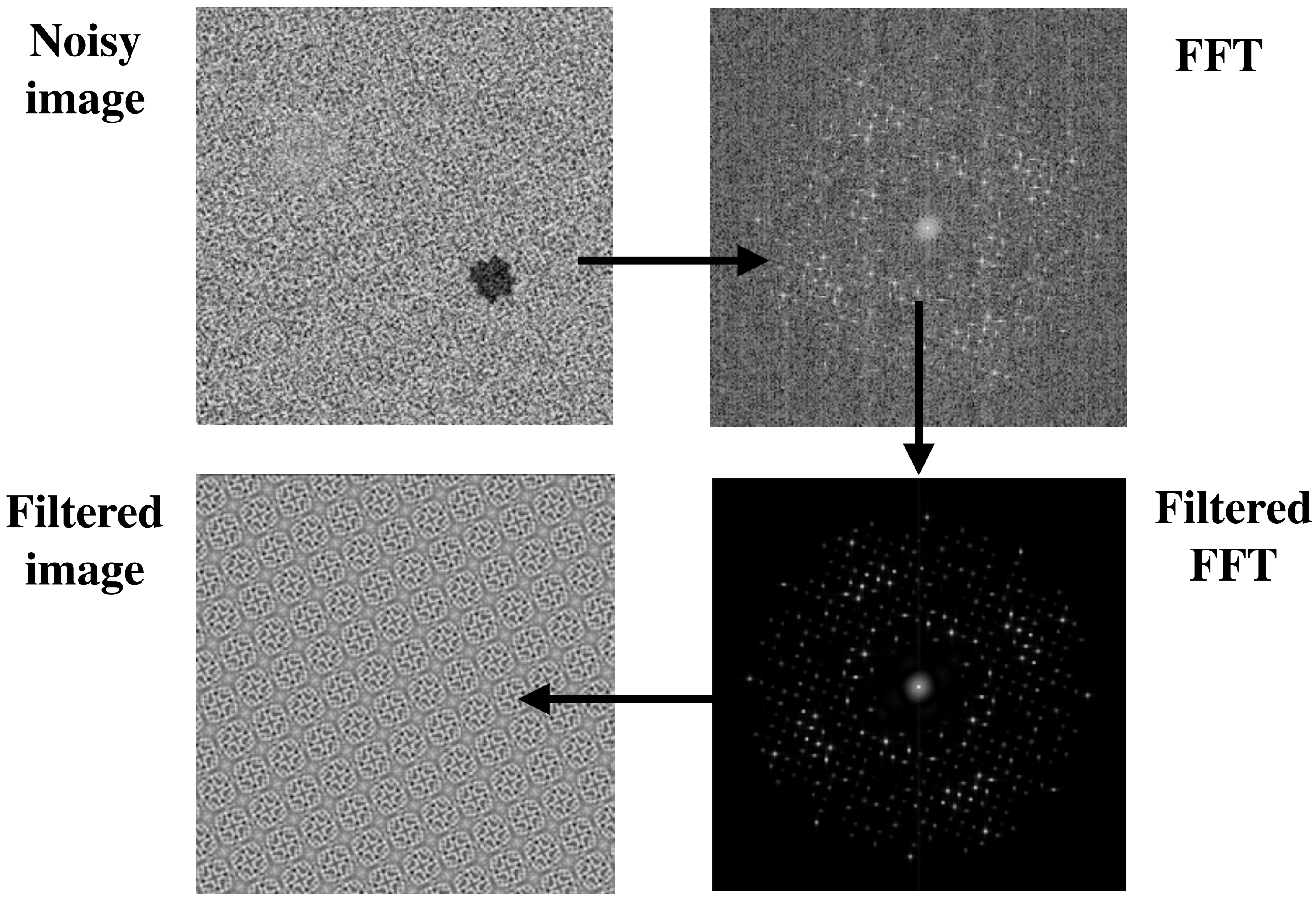
Real Space



Fourier Transform



Extract the information from the noise



Fourier Transformation

Fourier transformation.

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x} \, dx$$

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot \left[\cos(2\pi \cdot u \cdot x) - i \cdot \sin(2\pi \cdot u \cdot x) \right] dx$$

$$F(u) = FT\left(f(x)\right)$$

Inverse Fourier transformation exists.

$$f(x) = FT^{-1}\left(F(u)\right)$$

$$f(x) = \frac{1}{2\pi} \int F(u) \cdot e^{+i \cdot 2\pi \cdot u \cdot x} \, du$$

Fourier Equations

$$FT(a \cdot f(x)) = a \cdot F(u)$$

If you put more contrast in the image, then the FFT's amplitude gets stronger.

$$FT(f(x) + g(x)) = F(u) + G(u)$$

Adding two images f and g and calculating their FFT is like adding the FFTs F and G of them.

$$FT(f(ax)) = F(u / a)$$

If you stretch an image by a,
then you shorten the FFT by a.
(===> reciprocity)

$$FT(rotated \;\; f(x)) = rotated \;\; F(u)$$

If you rotate and image, then you also rotate its FFT.

Convolution

$$f(x) \otimes g(x) = FT^{-1}\left[F(u) \cdot G(u)\right]$$

Convolution of f with g in real space is slow. It can be done much faster by multiplying their FFTs, and calculating the inverse FFT of the result.

"Convolution of a set of spots with a duck produces a set of ducks."

"Convoluting a structure's projection map with a Point Spread Function gives you the recorded image. This is equivalent to multiplying the FFT of the map with the Contrast Transfer Function."
(=> deconvolution)

Cross-Correlation

$$f(x) \times g(x) = FT^{-1}\left[F(u) \cdot G^{*}(u)\right]$$

Couple Conjugation:

$G = a + ib$

$G^{*} = a - ib$

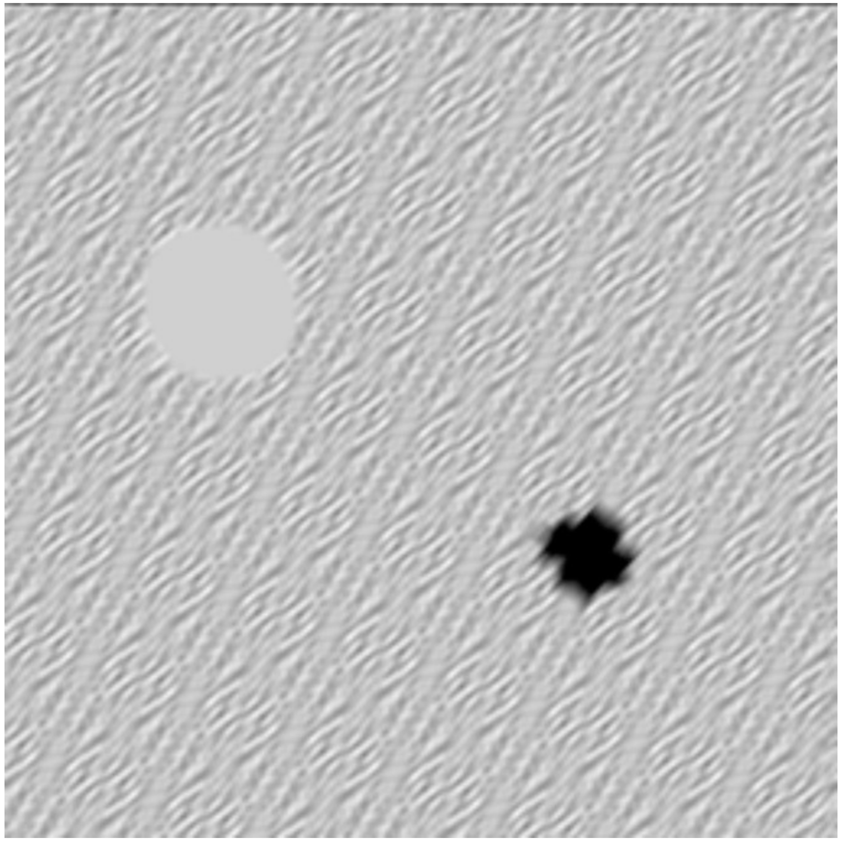
$Phase(G) = \Phi$

$Phase(G^{*}) = -\Phi$

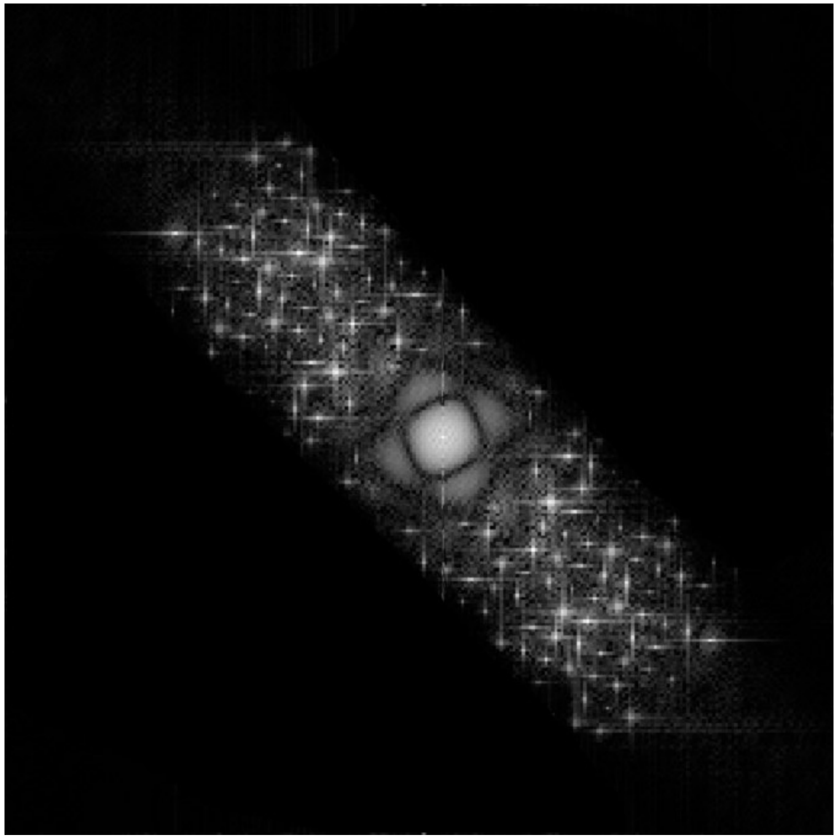
Cross-correlation of f with g in real space is slow. It can be done much faster by calculating their FFTs F and G, taking the complex conjugate of G*, multiplying F with G*, and calculating the inverse FFT of the result.

"Cross-correlation of a noisy image of many viruses with a virus-like circular reference produces a map with peaks that show where the viruses are."

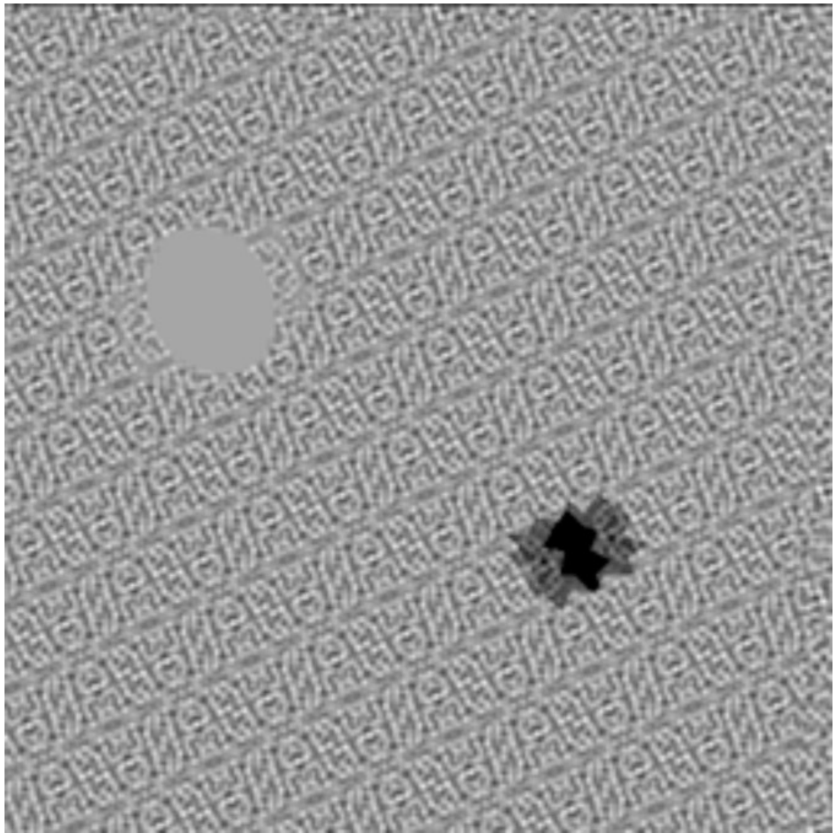
Real Space



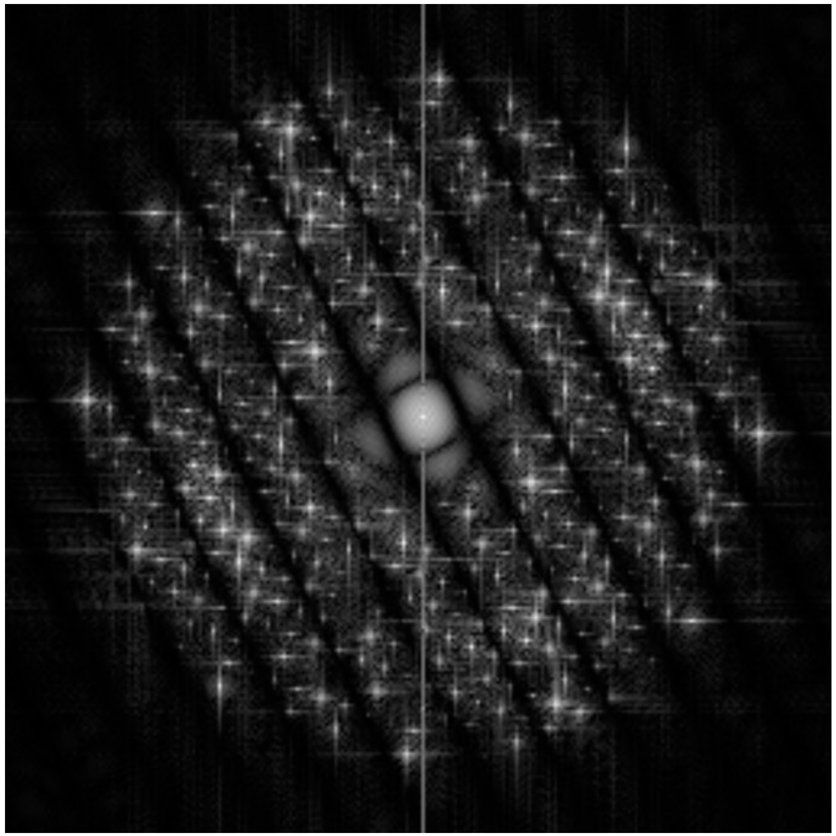
Fourier Transform



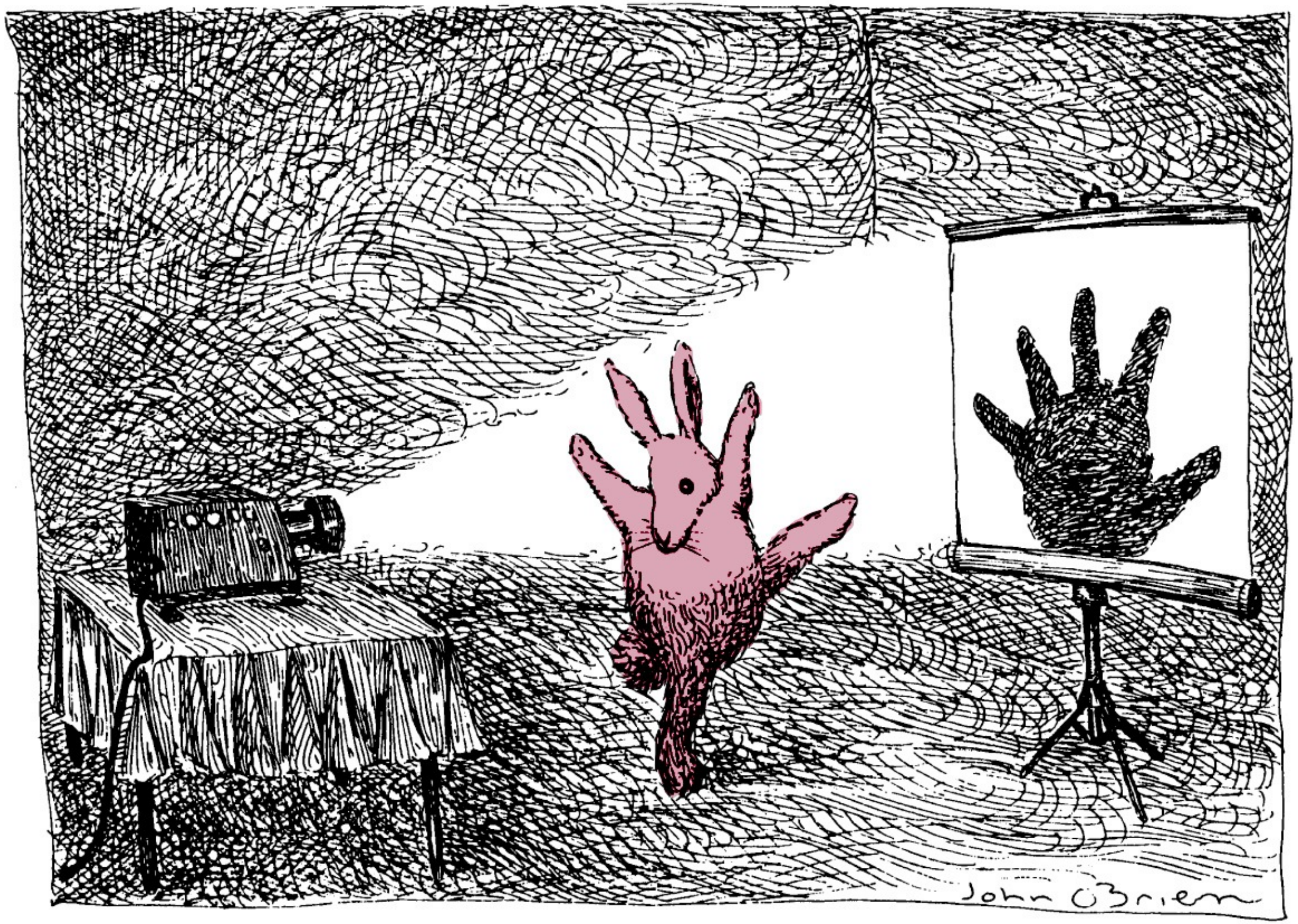
Real Space



Fourier Transform

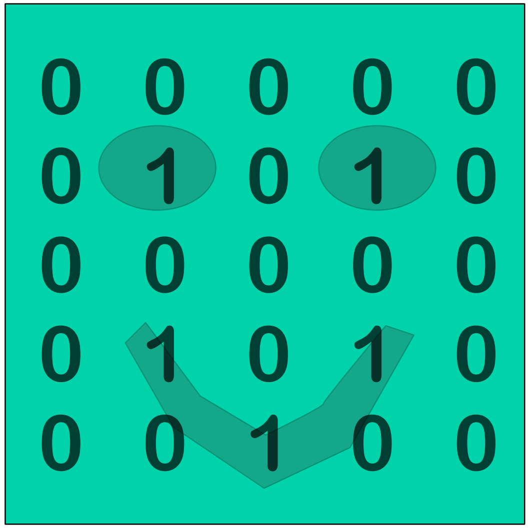


2D -> 3D



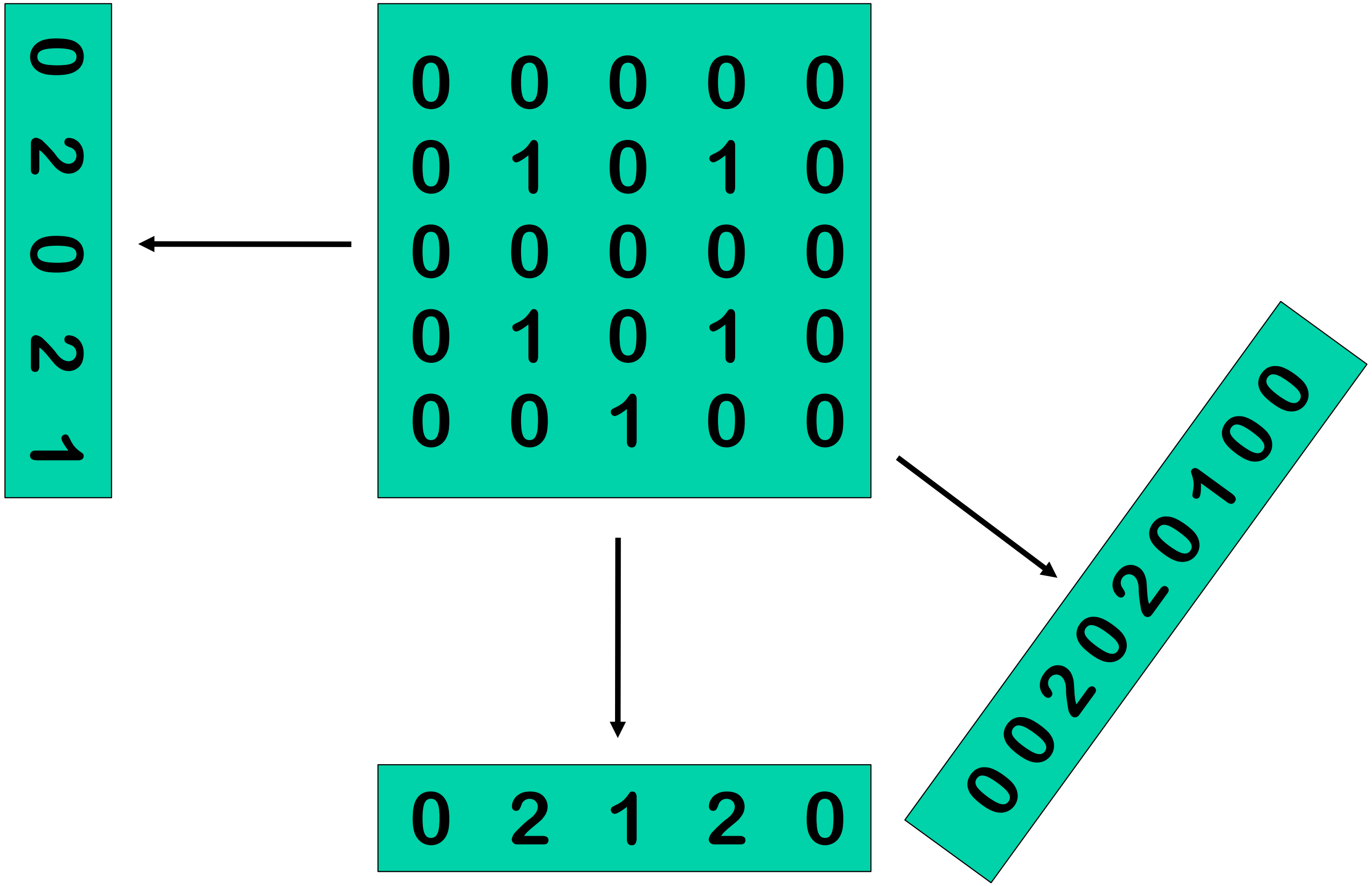
Drawing by John O‘Brien; © 1991 The New Yorker Magazine

Projection



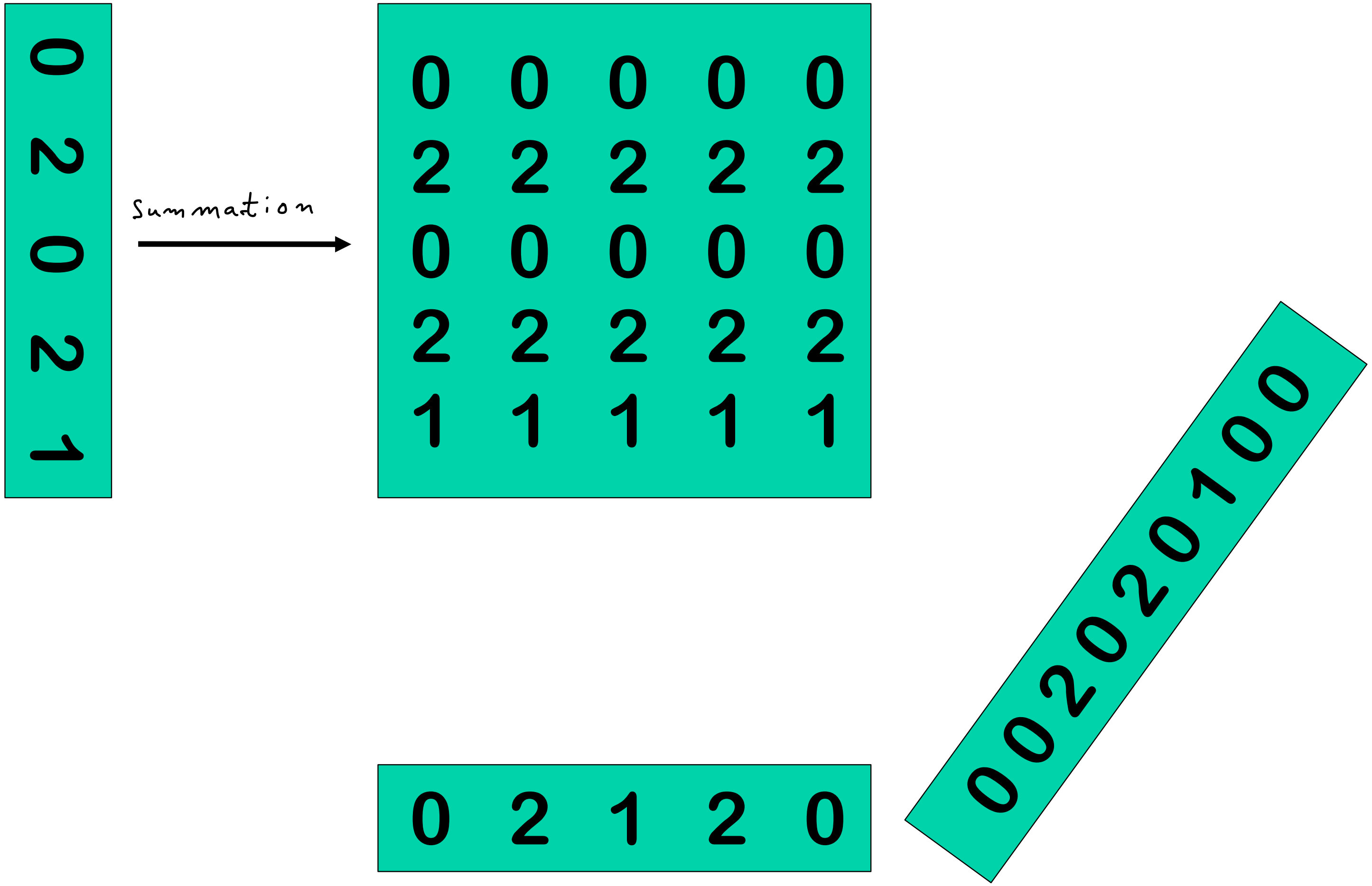
Real-Space Method

Projection



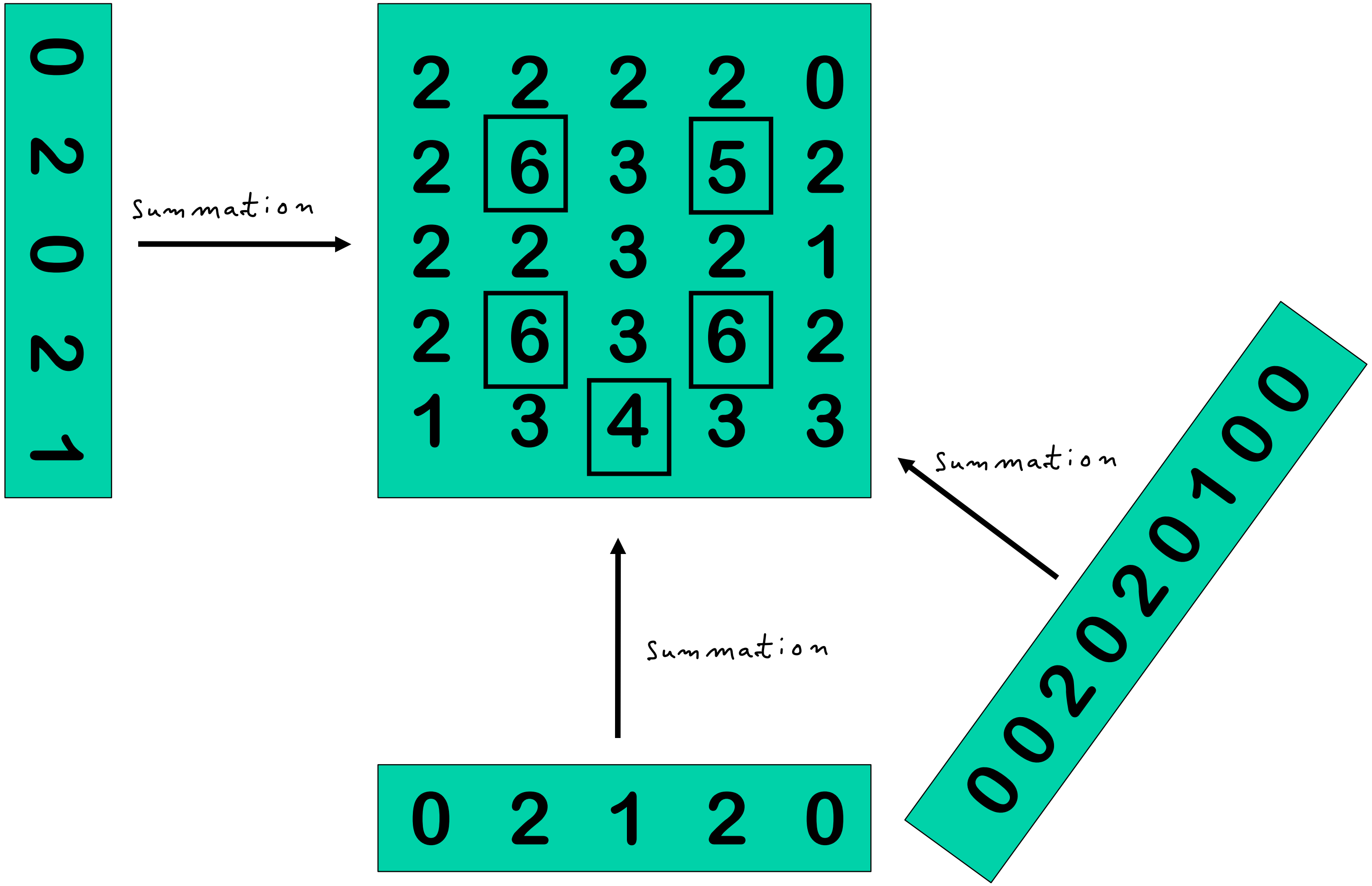
Real-Space Method

Backprojection



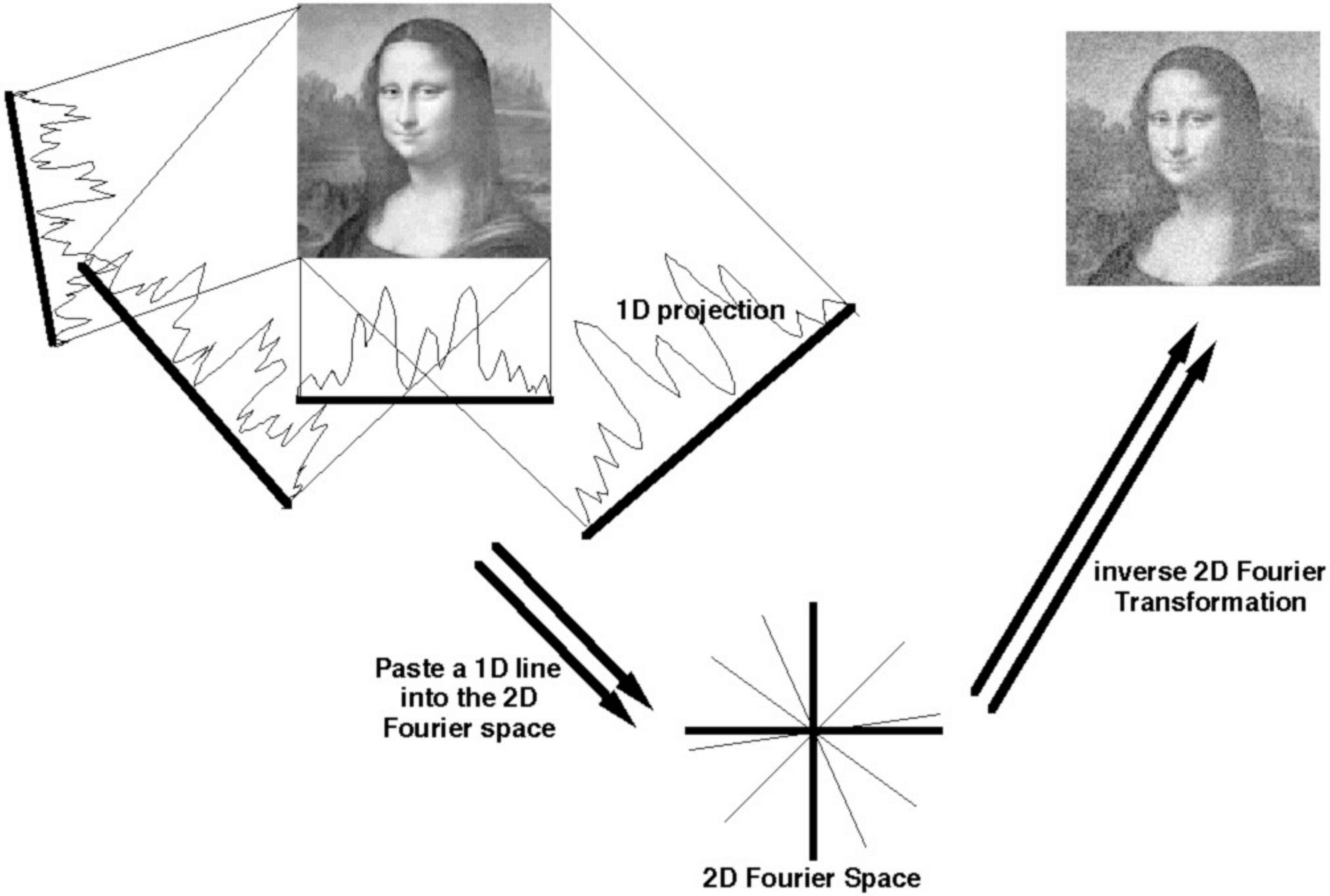
Real-Space Method

Backprojection



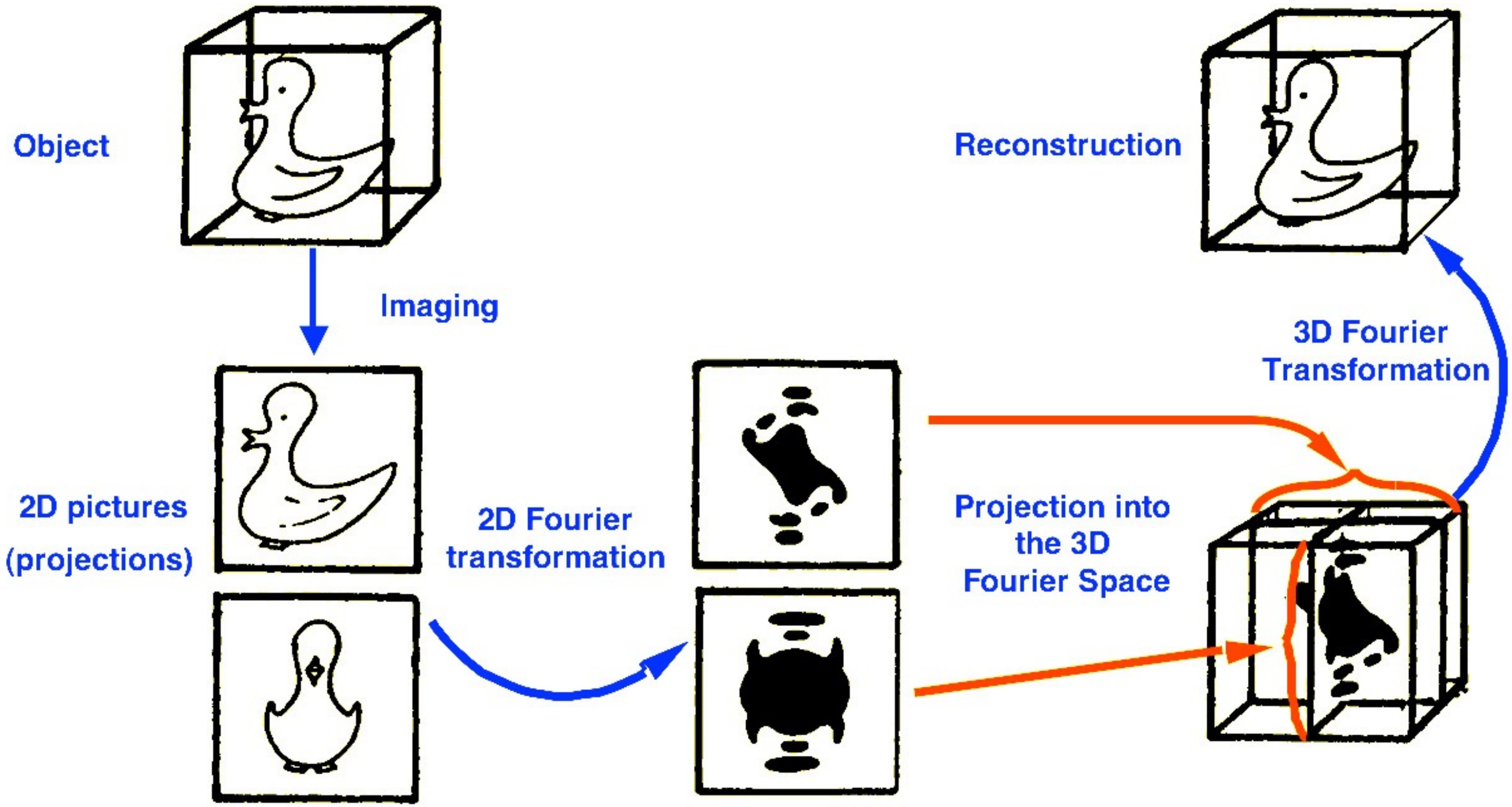
Real-Space Method

The Radon projection theorem for 3D reconstructions



Fourier-Space Method

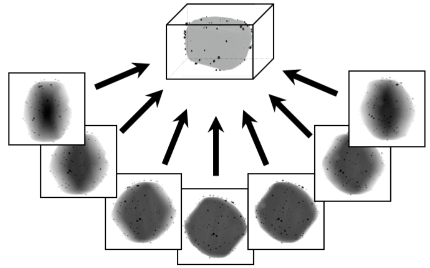
The Radon projection theorem for 3D reconstructions



Fourier-Space Method

Lake, J. (1971). In "Optical Transforms", (H. Lipson, Ed.), Academic Press, London.

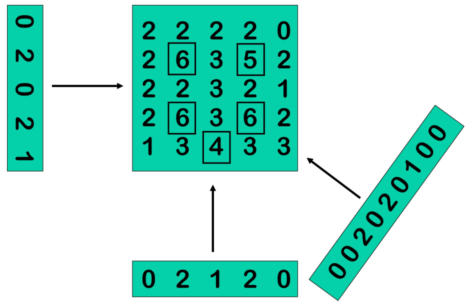
3D reconstruction



Computing requirements:

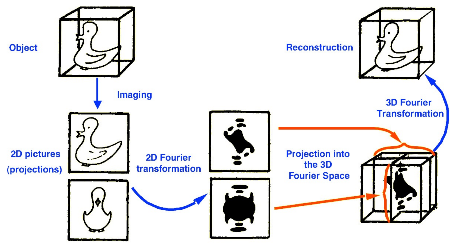
- 50 images at different tilt angles
- 1000x1000 px in each image
- 1000x1000x1000 voxels for the 3D reconstruction

Real-space backprojection:



Number of required summations:
50 x 1000 x 1000 x 1000 = 50’000’000’000

Fourier-space backprojection:



Lake, J. (1971). In "Optical Transforms", (H. Lipson, Ed.), Academic Press, London.

Number of required summations:
50 x 1000 x 1000 = 50’000’000

Central Section Theorem

$$F(u,v,w) = FT(f(x,y,z))$$

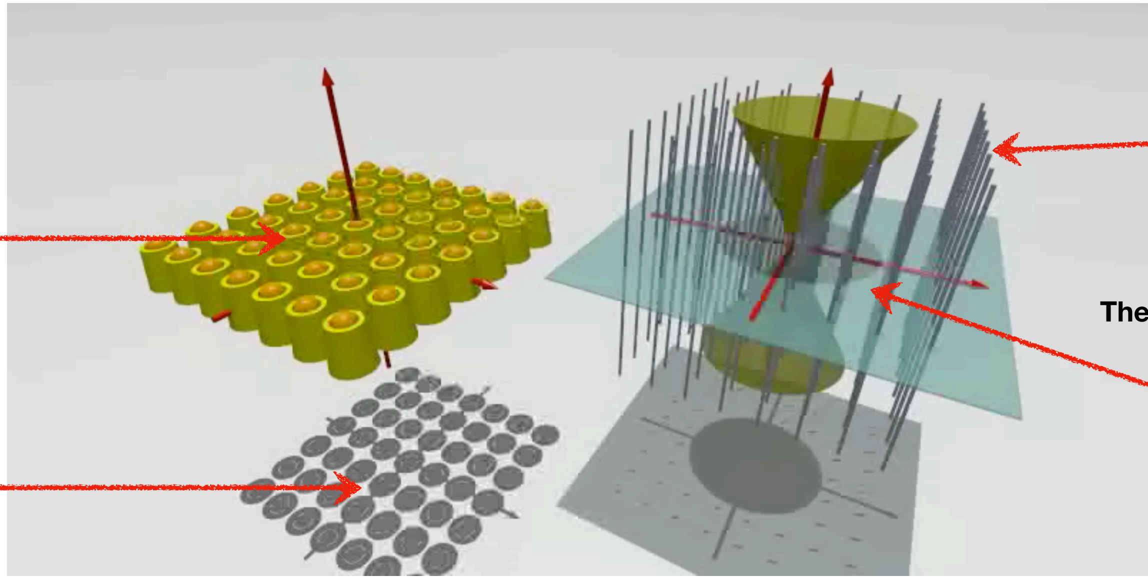
A 3D space with x,y,z corresponds to a Fourier space with u,v,w.

$$F(u,v,0) = FT\left[\int f(x,y,z)dz\right]$$

A projection in the vertical direction dz corresponds to the central section in the w=0 plane.

The Protein:
$f(x,y,z)$

The Projection Image:
$\int f(x,y,z)dz$



The Fourier Transform of the 3D reconstruction:
$F(u,v,w)$

The Fourier Transform of the Image:
$F(u,v,0)$

Special Cases

$$FT(rect(ax)) = \frac{1}{|a|} sinc\left(\frac{u}{a}\right)$$

The Fourier transform of a rectangular function is the sinc function.

$$FT(sinc(ax)) = \frac{1}{|a|} rect\left(\frac{u}{a}\right)$$

The Fourier transform of a sinc function is a rectangular function.

$$FT(e^{-ax^2}) = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{(\pi u)^2}{a}}$$

The Fourier transform of a Gaussian function is a Gaussian function.

$$FT(\delta(x)) = 1$$

The Fourier transform of a delta function is a constant.

$$FT(1) = \delta(u)$$

The Fourier transform of a constant function is a delta function.

Fourier Transformation

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x} \, dx$$

Use it to:

- *Analyze performance of microscope*
- *Analyze resolution and quality of image*
- *Modify or correct certain image artifacts (=> CTF)*

- *In case of crystal: Extract structure (=> Fourier filtering)*

- *Use to*
  Calculate cross-correlation function with a reference
  Calculate convolution with or deconvolution of a kernel function

- *In case of different sample orientations in set of images:*
  combine into 3D reconstruction (=> Backprojection)