1

Continuity equation: $\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0$

Note: $\underset{\sim}{x} \equiv \vec{x}$

Small perturbations in ρ & $\underset{\sim}{v}$:

$$\rho(\underset{\sim}{x}, t) = \rho_0(t) + \delta\rho(\underset{\sim}{x}, t)$$

$$\underset{\sim}{v}(\underset{\sim}{x}, t) = \underbrace{\underset{\sim}{v}_0(t)}_{\text{Hubble flow}} + \delta \underset{\sim}{v}(\underset{\sim}{x}, t)$$

in CE:

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \delta\rho}{\partial t} + \underset{\sim}{\nabla}\underbrace{\left[(\rho_0 + \delta\rho)(\underset{\sim}{v}_0 + \delta \underset{\sim}{v}_0)\right]}_{\underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{v}_0 + \rho_0 \delta \underset{\sim}{v} + \delta\rho \underset{\sim}{v}_0)} = 0$$

$\delta\rho\, \delta \underset{\sim}{v}_0$ neglect 2nd order terms!

$$\underbrace{\frac{\partial \rho_0}{\partial t} + \frac{\partial \delta\rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{v}_0)}_{=0 \text{ (CE for homogen. quantity)}} + \rho_0 \underset{\sim}{\nabla} \cdot \delta \underset{\sim}{v} + \underbrace{\delta \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \rho_0}_{=0 \text{ as } \rho_0(t)} +$$

$$+ \underbrace{\delta\rho \underset{\sim}{\nabla} \cdot \underset{\sim}{v}_0}_{\delta\rho\, 3H \text{ as } \underset{\sim}{v}_0 = H\underset{\sim}{x}} + \underset{\sim}{v}_0 \cdot \underset{\sim}{\nabla} \delta\rho = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \underset{\sim}{\nabla} \cdot \delta \underset{\sim}{v} + \delta\rho\, 3H + \underset{\sim}{v}_0 \cdot \underset{\sim}{\nabla} \delta\rho = 0 \quad \Big| : \rho_0$$

$D := \frac{\delta\rho}{\rho_0}$ Density contrast overdensity rel. to ave. homog.

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2

One can show that:

$$\frac{1}{\rho_0}\frac{\partial \delta\rho}{\partial t} = \frac{\partial D}{\partial t} - 3HD$$

$$\Rightarrow \frac{\partial D}{\partial t} - \cancel{3HD} + \underset{\sim}{\nabla}\cdot\delta\underset{\sim}{v} + \cancel{3HD} + v_0\,\underset{\sim}{\nabla}D = 0$$

"$\delta\rho/\rho_0$ allowed as ρ_0 const!"

$$\Rightarrow \boxed{\frac{\partial D}{\partial t} + \underset{\sim}{v_0}\cdot\underset{\sim}{\nabla}D + \underset{\sim}{\nabla}\cdot\delta\underset{\sim}{v} = 0} \quad (1)$$

To further simplify → move to co-moving coordinates $\underset{\sim}{r}$ & $\underset{\sim}{u}$, coordinate system moving with Hubble expansion

Let's express physical coordinates x, v in co-moving units $\underset{\sim}{r}, \underset{\sim}{u}$

$\underset{\sim}{x} = a\underset{\sim}{r}$; $\quad \underset{\sim}{\nabla}_x \longrightarrow \frac{\nabla_r}{a}$

$\underset{\sim}{\nabla}_x\underset{\sim}{x} = 3$ to keep equality with $\underset{\sim}{\nabla}\, a\underset{\sim}{r}$ ↳ $\frac{\underset{\sim}{\nabla}_r}{a}$; $a\frac{\underset{\sim}{\nabla}_r}{a}\underset{\sim}{r} = 3$

$\underset{\sim}{v} = \dot{\underset{\sim}{x}} = \dot{a}\underset{\sim}{r} + a\dot{\underset{\sim}{r}} = \underset{\sim}{v_0} + \delta\underset{\sim}{v}$

$\underset{\sim}{v_0}$: mean velocity Hubble expansion $\dot{a}\underset{\sim}{r} = \frac{\dot{a}}{a}\underset{\sim}{x}$

$\delta\underset{\sim}{v} = a\dot{\underset{\sim}{r}} = \underset{\sim}{u}$: perturbation in v ↳ peculiar velocity rel. to Hubble flow

Also time derivative changes moving into co-moving space (4D ~~time-space~~ space-time vectors)

$$\left.\frac{\partial}{\partial t}\right|_{\underset{\sim}{x}} \longrightarrow \left.\frac{\partial}{\partial t}\right|_{\underset{\sim}{r}} - \underbrace{\frac{\dot{a}}{a}}_{H}\underset{\sim}{r}\cdot\underset{\sim}{\nabla}_r$$

Insert into (1):

$$\frac{\partial D}{\partial t} - \frac{\ddot{a}}{a}\,\underset{\sim}{r}\cdot\underset{\sim}{\nabla}_r D + \overset{u_0}{\dot{a}\,\underset{\sim}{r}}\cdot\frac{\underset{\sim}{\nabla}_r}{a} D + \frac{\underset{\sim}{\nabla}_r}{a}\cdot\overset{\delta u}{\underset{\sim}{u}} = 0$$

cancel!

$$\Longrightarrow \boxed{\frac{\partial D}{\partial t} + \frac{\underset{\sim}{\nabla}_r}{a}\cdot\underset{\sim}{u} = 0}$$

Perturbed continuity equation if coordinate system is moving with Hubble flow

dependence on density contrast & peculiar velocity

simpler than in physical coordinates...