2

Quantum Mechanics

As mentioned in the introduction, this book assumes some basic understanding of quantum mechanics at the undergraduate level. This understanding is essential for many reasons: not only to be able to write down realistic models, compute things, and make predictions, but also because quantum physics is a world full of subtleties and unexpected phenomena which may cause major confusion and headaches when seen for the first time.

Consequently, here is a friendly warning: If you never took a course on quantum physics and the formalism of quantum mechanics, take it, or grab some books that can help you overcome this initial barrier. There are many good books to choose from, such as the two volumes of Cohen-Tannoudji et al. (1977) or the wonderful and complete book by Ballentine (1998). Simply take the one that resonates best with you.

This said, I felt the need to write down a chapter where I could summarize many of the concepts and tools from quantum mechanics that are repeatedly invoked in this book. Some of these are pretty basic, such as the relation between the Schrödinger equation and unitary operators, but others are a bit more subtle and extend beyond a typical course on quantum mechanics, such as the notion of density matrices and master equations. Please take this therefore as a "unifying" chapter that provides a common language and definitions, and which you may freely skip if you have a graduate or postgraduate level in the study of quantum physical systems.

2.1 Canonical Quantization

Quantum physics is an old field of research, whose birth we attribute to Planck's theory for the black-body radiation. A black body is an ideal object that can absorb energy at any frequency of the spectrum, which incidentally implies that it is the body that can emit energy most efficiently at any given frequency. In 1900, the

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physicist Max Planck showed that experiments measuring the radiation from black bodies could be explained by assuming that these objects – which at the time were just perfect cavities with a tiny hole – could only exchange energy in fixed amounts or *quanta*, determined by the frequency ν of the light emitted or absorbed by the black cavity.

In his treatment, Planck models the excitations of the black body as a collection of harmonic oscillators with frequencies that cover the measured spectrum – i.e., $\nu(\mathbf{k})$, labeled by wave vectors $\mathbf{k}$. The energy of those oscillators is quantized, which means that the oscillators equilibrate to the same temperature by exchanging discrete units of energy or *quanta* with the environment. If we could measure the state of the black body, its energy would be a sum of the *quanta* $n_{\mathbf{k}}$ that are stored in each electromagnetic mode $\mathbf{k}$:

$$E = \sum_{\mathbf{k}} h\nu(\mathbf{k}) \times n_{\mathbf{k}}, \quad n_{\mathbf{k}} \in \{0, 1, 2, \ldots\}. \tag{2.1}$$

In this model, each oscillator has associated a quantum of energy $h\nu(\mathbf{k})$ determined by the frequency[1] $\nu(\mathbf{k})$ and Planck's constant $h \simeq 6.62607004(81) \times 10^{-34}$ J/Hz. The collection of all integers $|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\rangle$ is a unique configuration of the black body, which we call the *quantum state*, and the collection of all states is used to develop a statistical model of the black body's spectrum.

Despite his success, Planck was very wary of extending the idea of quanta to the actual electromagnetic field. It was Einstein who made the connection between Planck's quanta and the existence of a particle of light, the *photon*. With this particle, Einstein could explain in 1905 the photoelectric effect: Some materials may convert light into an electrical current, but when the intensity of light is lowered enough, this current becomes a series of discrete random bursts, which Einstein associated with the absorption of photons. This successful explanation was shortly followed by Bohr and Rutherford's model of the atom, based on quantized electronic orbits that explained the discrete spectra of light-emitting atoms. Barely a decade later, Schrödinger (1926) and Heisenberg (1925) replaced all ad hoc quantization ideas with two equivalent formulations of quantum mechanics based on wave and matrix equations. In both theories, the discretization of energies is a mathematical consequence of the discrete spectra of the operators that govern the evolution of light and matter. Put to work, the newly born theory provided quantitative explanations for the spectra of atoms, molecules, solids, and the electromagnetic field itself, consolidating nonrelativistic quantum mechanics as

[1] Ordinary frequencies are typically denoted by the letter ν and are measured in the S.I. unit of Hertz (Hz). Quite often we will also use angular frequencies $\omega = 2\pi\nu$, sometimes denoted in rad/s or s^{-1}. In the first case, the quantum of energy is given by $h\nu$, while in the second case it is given by Planck's reduced constant $\hbar\omega$ with $\hbar = h/2\pi$.

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the tool for understanding chemistry, solid-state physics, nano-electronics, and photonic devices, to name a few examples.

Schödinger's and Heisenberg's formulations are powerful theories that explain the microscopic behavior of Nature in a bottom-up fashion: starting from elementary components – the electron, the proton, and the neutron – and fundamental interactions – for instance, the Coulomb attraction between protons and electrons mediated by the electromagnetic field – one builds a many-body equation whose solution accounts for all the physics we observe in the laboratory. Unfortunately, the bottom-up approach does not always scale well as we move on to larger systems. In this book, we are concerned with solid-state superconducting devices that include more than 10^{24} atoms, all collectively exhibiting quantum mechanical phenomena. It is unfathomable to even think of writing an equation for all those particles, and we are forced to seek effective descriptions that are consistent with the principles and rules of the underlying quantum mechanical theory.

Shortly after the publication of Schrödinger's and Heisenberg's work, Paul Dirac developed an alternative derivation of quantum mechanics, known as *canonical quantization*, that establishes a link between the quantum model for a given object (particle, field, etc.) and the dynamics that we would expect from it in a classical world. In Chapter 4, we will apply this procedure to the quantization of an electrical circuit, developing a quantum theory of superconducting circuits. This theory will be consistent with the microscopic description introduced in Chapter 3, and it will provide the appropriate limit of the circuit when temperatures are high enough that superconductivity is lost, or quantum phenomena are masked.

2.1.1 Hamiltonian Equations

For simplicity, we will describe how canonical quantization works for a simple object: a point-like particle[2] with position $\mathbf{x}$ and momentum $\mathbf{p} = m\dot{\mathbf{x}}$, moving in an external potential $V(\mathbf{x})$. The particle's trajectory is governed by a set ordinary differential equations – Newton's equations – which we write in terms of the particle's acceleration $\ddot{\mathbf{x}}$ and the force $\nabla V(\mathbf{x})$ experienced by the particle

$$\ddot{\mathbf{x}} = -\frac{1}{m}\nabla V(\mathbf{x}). \tag{2.2}$$

Newton's equation can be derived from a *stationary action* principle, as the trajectory that minimizes the action $S = \int_{t_1}^{t_2} \mathcal{L}(\dot{\mathbf{x}}, \mathbf{x})\mathrm{d}t$. The functional S maps orbits $x(t)$ to real numbers according to the *Lagrangian*

[2] As we will see in Chapter 4, this is not a futile exercise, because the harmonic potential $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$ is formally analogous to the simplest electrical circuit, an LC resonator, and describes how this circuit is actually quantized.

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$$\mathcal{L}(\dot{\mathbf{x}}, \mathbf{x}) = \frac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x}). \tag{2.3}$$

According to the stationary principle, a small perturbation of the particle's true trajectory $x_\varepsilon(t) = x(t) + \varepsilon(t)$ should leave the action unperturbed up to second-order corrections $S[x_\varepsilon] = S[x] + \mathcal{O}(\varepsilon^2)$. This stationary principle produces Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}_n} = \frac{\partial \mathcal{L}}{\partial x_n}, \tag{2.4}$$

which are equivalent to the original Newtonian equations (2.2).

We now introduce a *Hamiltonian formulation*, where the functional that generates the dynamical equations is a function of two canonically conjugate variables, $\mathbf{x}$ and $\mathbf{p}$, with the prescription

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}\dot{\mathbf{x}} - \mathcal{L}(\dot{\mathbf{x}}, \mathbf{x}), \quad \text{with } \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}. \tag{2.5}$$

This *Legendre transform* establishes a link between the particle's velocity $\dot{\mathbf{x}}$ and its canonical momentum $\mathbf{p}$, which now replaces the former in all equations. The transform also produces an object, the *Hamiltonian* $H(\mathbf{x}, \mathbf{p})$, governing the orbits of the particle. More precisely, any observable $O(\mathbf{x}, \mathbf{p}, t)$ that we can construct as a function of the canonical variables and time evolves according to the Hamiltonian equation

$$\frac{\mathrm{d}}{\mathrm{d}t} O = \{O, H\} + \frac{\partial O}{\partial t}, \tag{2.6}$$

with the classical *Poisson brackets*

$$\{A, B\} = \sum_j \left(\frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial x_i} \frac{\partial A}{\partial p_i} \right). \tag{2.7}$$

In particular, since $\{x_i, p_j\} = \delta_{ij}$ this prescription trivially recovers Newton's equations, but now expressed as a set of first-order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{x} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m} \mathbf{p}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{p} = -\frac{\partial H}{\partial \mathbf{x}} = -\nabla V(\mathbf{x}). \tag{2.8}$$

2.1.2 Quantum Observables

Dirac's canonical quantization describes the transition from a Hamiltonian theory of a classical particle to a quantum mechanical theory that preserves the particle's dynamical equations (2.6). First of all, following the axioms of quantum mechanics, it introduces a Hilbert space of *vector states* describing our system. Next, it replaces any measurable quantity, including $\mathbf{x}$, $\mathbf{p}$, and any other function thereof $O(\mathbf{x}, \mathbf{p})$, with

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linear Hermitian operators $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$, $\hat{O}$ acting on this Hilbert space. Each observable will have a spectrum of eigenvalues and eigenstates determining all possible measurement outcomes.

In our toy model, the quantum state of a particle that is at position $\mathbf{r} \in \mathbb{R}^N$ is associated a vector $|\mathbf{r}\rangle$ in the Hilbert space, with the property $\hat{x}_n |\mathbf{r}\rangle = r_n |\mathbf{r}\rangle$. Generic states are constructed as quantum superpositions of different measurement outcomes, such as the wavefunction $\psi(\mathbf{r})$:

$$|\psi\rangle = \int \psi(\mathbf{r}) |\mathbf{r}\rangle \, \mathrm{d}^N\mathbf{r}. \tag{2.9}$$

The weights of the wavefunction $\psi(\mathbf{r}) \in \mathbb{C}$ are complex numbers whose modulus gives the probability distribution $P(\mathbf{r}) = |\psi(\mathbf{r})|^2$ that the particle is found at the position $\mathbf{r}$, if the observable $\hat{\mathbf{x}}$ is ever measured. States are normalized, so that the total probability adds up to one, $\langle \mathbb{1} \rangle_\psi = \langle \psi | \psi \rangle = \int |\psi(\mathbf{r})|^2 \mathrm{d}\mathbf{r} = 1$, and we can define the *expectation values* of measurements:

$$\begin{aligned} \langle \hat{\mathbf{x}} \rangle_\psi = \langle \psi | \hat{\mathbf{x}} | \psi \rangle &= \iint \psi(\mathbf{r}_0)^* \psi(\mathbf{r}_1) \langle \mathbf{r}_0 | \, \hat{\mathbf{x}} \, | \mathbf{r}_1 \rangle \, \mathrm{d}^N\mathbf{r}_0 \mathrm{d}^N\mathbf{r}_1 \\ &= \int \mathbf{r}_0 |\psi(\mathbf{r}_0)|^2 \mathrm{d}\mathbf{r}_0. \end{aligned} \tag{2.10}$$

Note how, by using the orthogonality of position eigenstates $\langle \mathbf{r}_0 | \hat{\mathbf{x}} | \mathbf{r}_1 \rangle = \mathbf{r}_1 \langle \mathbf{r}_0 | \mathbf{r}_1 \rangle = \mathbf{r}_1 \delta(\mathbf{r}_0 - \mathbf{r}_1)$, we recovered the formula for the average over the probability distribution $P(\mathbf{r})$.

Canonical quantization includes one final prescription that makes the algebra of operators and states consistent with the classical limit of these equations. We replace everywhere the Poisson brackets for classical variables with the commutator between the respective observables $\{A, B\} \to -i[\hat{A}, \hat{B}]/\hbar$, where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. For the isolated particle, this prescription transforms $\{x_n, p_m\} = \delta_{nm}$ into

$$[\hat{x}_n, \ \hat{p}_m] = i\hbar\delta_{nmj}. \tag{2.11}$$

If our space of positions is continuous and contained in the region $\Omega \subset \mathbb{R}^N$, we can build our Hilbert space using integrable functions $\psi(\mathbf{r}) \in L^2(\Omega)$, associating position and momentum with operators $\hat{x}_n \psi(\mathbf{r}) = r_n \psi(\mathbf{r})$ and $\hat{p}_n \psi(\mathbf{r}) = -i\hbar \partial_{r_n} \psi(\mathbf{r})$ that satisfy the commutation relations (2.11).

Irrespective of the implementation of our Hilbert space, (2.11) implies that canonically conjugate operators are incompatible: They cannot share a common basis of eigenstates. Thus, a state with a well-defined position $|x_0\rangle$ cannot have a well-defined value of the momentum. This leads to the *Heisenberg uncertainty principle*:

$$\Delta x_n \Delta p_m \geq \frac{1}{2}\hbar\delta_{nm}, \tag{2.12}$$

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which relates the variances of pairs of observables, $\Delta x_n = \sqrt{\langle \hat{x}_n^2 \rangle - \langle \hat{x}_n \rangle^2}$ and Δp_m. As we will see later, this uncertainty has also a physical manifestation in the world of quantum circuits, where position and momenta are replaced by voltage and intensity, quantities that cannot be measured simultaneously with absolute precision.

2.1.3 Unitary Evolution

We have introduced quantum states and observables as two mathematical objects that together predict the statistics of measurement outcomes. This information is bound to change in time, as observables and states evolve. In canonical quantization, the identification of Poisson brackets with commutators translates the classical equation (2.6) into the *Heisenberg equation*:

$$\frac{\mathrm{d}\hat{O}}{\mathrm{d}t} = -\frac{i}{\hbar}[\hat{O}, \hat{H}] + \frac{\partial \hat{O}}{\partial t}. \tag{2.13}$$

In this model, the dynamics is generated by a Hamiltonian operator that results from replacing the canonical variables with the corresponding observables $\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 + V(\hat{\mathbf{x}})$. In the Heisenberg picture, observables change in time starting from a well-known initial condition $O(t_0) = O_0$. The states $|\psi_0\rangle$ remain stationary, and they are regarded as objects that map the changing observables to their expectation values $\bar{O}(t) = \langle \hat{O}(t) \rangle = \langle \psi_0 | \hat{O}(t) | \psi_0 \rangle$.

The Heisenberg equation is rather inconvenient: We have to work with big and complex operators, and extracting the measurement statistics becomes a very convoluted process. In many situations, we would rather work with an equation that determines how states evolve from, say, an initially localized configuration $\psi_0(x) = \delta(x - x_0)$, spreading to other meaurement outcomes. This information is provided by the *Schrödinger equation* or *Schrödinger picture*, whereby observables have an immutable representation, but wavefunctions change in time $\bar{O}(t) = \langle \hat{O} \rangle_{\psi(t)} = \langle \psi(t) | \hat{O}_0 | \psi(t) \rangle$, with

$$i\hbar \partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle , \quad \text{with } |\psi(t_0)\rangle = |\psi_0\rangle . \tag{2.14}$$

For our isolated particle in an external potential $V(\mathbf{x})$, using the position representation, where $\hat{\mathbf{p}} = -i\hbar\nabla$, this results into a simple wave equation for the complex amplitude of probability $\psi(\mathbf{x})$:

$$i\hbar \partial_t \psi(\mathbf{x}) = \left[\frac{1}{2m}(-i\hbar\nabla)^2 + V(\mathbf{x}) \right] \psi(\mathbf{x}). \tag{2.15}$$

Even if they look very different, the Schrödinger and Heisenberg representations give the same predictions because they are both solved by a common unitary

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transformation called the *evolution operator*. This operator is the solution of an enlarged Schrödinger equation:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t,t_0) = \hat{H}\hat{U}(t,t_0), \quad \text{with } \hat{U}(t_0,t_0) = \mathbb{1}. \tag{2.16}$$

In the case of constant Hamiltonians, the unitary operator U is a Lie rotation in the Hilbert space, generated by the Hamiltonian:

$$\hat{U}(t,t_0) = \exp[-i(t-t_0)H/\hbar]. \tag{2.17}$$

This operator is unitary $UU^\dagger = U^\dagger U = \mathbb{1}$. It can be inverted $\hat{U}(t_2,t_1)^{-1} = \hat{U}(t_1,t_2)$ and solves both the Schrödinger $|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi_0\rangle$ and the Heisenberg equations $\hat{O}(t) = \hat{U}(t_0,t)\hat{O}_0\hat{U}(t,t_0)$, as mentioned before.

2.2 Two-Level Systems

Not all physical systems have continuous degrees of freedom. We are going to work with smaller systems that only have two or three configurations that are active in a given experiment. These discrete systems have smaller Hilbert spaces, with wavefunctions defined in complex vector spaces. For a quantum system with two possible states $|0\rangle$ and $|1\rangle$, the wavefunctions in the two-dimensional Hilbert space are described by two complex amplitudes:

$$|\psi\rangle = \psi_0\,|0\rangle + \psi_1\,|1\rangle \;\leftrightarrow \boldsymbol{\Psi} = \begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix} \in \mathcal{H} = \mathbb{C}^2, \tag{2.18}$$

with the usual normalization $|\psi_0|^2 + |\psi_1|^2 = 1$.

Two-dimensional Hilbert spaces are very common. They are a natural representation of the spin $s = 1/2$ states of an electron, a proton, or a neutron: $|0\rangle$ and $|1\rangle$ correspond to spin down and up along a given direction; they are sometimes used for describing the polarization states of a photon, horizontal versus vertical; and they appear most frequently in quantum optics when modeling atomic transitions – i.e., ground state $|0\rangle$ versus excited state $|1\rangle$ – and light–matter interaction. Nowadays, two-dimensional quantum systems are also called *qubits*, because, in analogy to the classical bit, they represent the minimal quantum object where information can be stored and processed. We show in Chapter 6 that it is possible to build superconducting circuits that are accurately described as qubits, and discuss in later chapters how these circuits are applied to quantum computing and simulation.

The algebra of two-level systems is analyzed using a complete set of observables, called the Pauli matrices:

$$\sigma^x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma^y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.19}$$

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Note that in this representation, $|0\rangle$ and $|1\rangle$ are the two eigenstates of σ_z, and we write $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. If we enlarge the set of Pauli matrices to include the identity, $\sigma^\alpha|_{\alpha=0}^3 = \{\mathbb{1}, \sigma^x, \sigma^y, \sigma^z\}$, we have a basis where we can expand any observable in this Hilbert space:

$$\hat{O} = \frac{1}{4}\sum_{\alpha=0}^{3} \mathrm{tr}\big(\hat{O}\hat{\sigma}^\alpha\big)\hat{\sigma}^\alpha. \tag{2.20}$$

In particular, a general qubit Hamiltonian has the form

$$\hat{H} = E + B\,\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}, \tag{2.21}$$

where $\mathbf{n}$ is a direction in the three-dimensional space, $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ and E and B are constants. For spins, the Hamiltonian is interpreted as the coupling between the qubit's dipole moment $\propto \hat{\boldsymbol{\sigma}}$ and the magnetic field $B\mathbf{n}$ along the given direction.

Interestingly, this form has two important properties. First, because $\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}$ has the properties of a Pauli matrix, the eigenvalues of this Hamiltonian are simply $\lambda_\pm = E \pm B$. Second, and for the same reasons, we can use the Pauli expansion to compute the evolution operator of the qubit:

$$\hat{U}(t,0) = \exp\left(-\frac{it\hat{H}}{\hbar}\right) = e^{-iEt/\hbar}\left[\cos(Bt)\,\mathbb{1} - i\sin(Bt)\,\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}\right]. \tag{2.22}$$

We will use this formula when studying superconducting qubits and the implementation of single-qubit gates in Section 6.1.4.

2.3 Density Matrices

The formalism of unitary evolution for a quantum system assumes that the system under study is perfectly isolated from other quantum or classical objects and subject to error-free control. Even if systematic errors are greatly reduced, no physical system can be perfectly isolated: At the very least, there will always be the omnipresent electromagnetic field, carrying the cosmic background radiation and putting our system in contact with the noisy classical world.

It is therefore safe to say that experiments never prepare pure states $|\psi\rangle$. Instead, real quantum systems must be described using an ensemble operator, also known as *density matrix* $\hat{\rho}$: a nonnegative Hermitian operator $\hat{\rho} = \hat{\rho}^\dagger \geq 0$, with proper normalization $\mathrm{tr}(\hat{\rho}) = 1 (\sim \sum_n \langle n|\hat{\rho}|n\rangle)$, which gives expectation values of operators as $\langle\hat{O}\rangle = \mathrm{tr}(\hat{O}\hat{\rho}) \sim \sum_n \rho_{nn}\langle n|O|n\rangle$.

Pure states $|\psi_\chi\rangle$ written as density matrices become projectors $\hat{\rho}_\chi = |\psi_\chi\rangle\langle\psi_\chi|$. General states, however, are *mixed states,* because they can be reconstructed as

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classical ensembles of pure states created with different classical probabilities $p(\chi)$ (Ballentine, 1970):

$$\hat{\rho} = \int |\psi_\chi\rangle\langle\psi_\chi| \, p(\chi) \mathrm{d}\chi . \tag{2.23}$$

This equation (2.23) describes the output of an experiment where parameters have some uncertainty. Mixed states can also arise when a system enters in contact with another system, called the environment. In principle, we should consider the global wavefunction of the system plus its environment, allowing both to be correlated $|\Psi_{\text{global}}\rangle = \sum_{s,E} \Psi_{s,E} \, |s\rangle \otimes |E\rangle \in \mathcal{H}_{\text{system}} \otimes \mathcal{H}_{\text{environment}}$. However, since we will not have access to all degrees of freedom of the environment, we must trace out all the information that we ignore, obtaining a much smaller ensemble that only describes our system:

$$\hat{\rho}_{\text{sys}} = \text{tr}_{\text{environment}} \, |\Psi_{\text{global}}\rangle \, \langle\Psi_{\text{global}}| = \sum_E \sum_{s,s'} \Psi_{s,E} \Psi^*_{s',E} \, |s\rangle\langle s'| \, . \tag{2.24}$$

Density matrices are particularly simple in the case of two-level systems, where they can be expanded in the qubit basis $\rho_{ij} := \langle i|\hat{\rho}|j\rangle$, or as a combination of Pauli operators (2.20):

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \sum_{i,j=0}^{1} \rho_{ij} \, |i\rangle\langle j| = \frac{1}{2}\mathbb{1} + \frac{1}{2} \sum_{\alpha=x,y,z} S_\alpha \hat{\sigma}^\alpha . \tag{2.25}$$

The set of all physical states with $\mathbf{S} = (S_x, S_y, S_z)$ falls inside the *Bloch sphere* $|\mathbf{S}| \leq 1$. The surface of this sphere is formed by pure state ($|\mathbf{S}| = 1$), and its center is the completely depolarized state $\mathbf{S} = 0$ or $\hat{\rho} = \frac{1}{2}\mathbb{1}$.

The completely depolarized state is an example of *classical state*, diagonal density matrices – $\rho_{10} = \rho^*_{01} = 0$ – which may be constructed as a convex combination $\hat{\rho} = \rho_{00} \, |0\rangle\langle 0| + \rho_{11} \, |1\rangle\langle 1|$ of preparing state $|0\rangle$ with probability $P_0 = \rho_{00}$ and preparing state $|1\rangle$ with probability $P_1 = 1 - P_0$. Compare this now with the *superposition state*, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. This state exists only in the quantum model for a two-level system. This particular superposition maximizes the off-diagonal elements ρ_{01} and ρ_{10}, also known as *coherences*, recognized as a true signature of quantumness in states.

Sometimes a quantum system is sufficiently isolated and the timescales of study are short enough that we can approximate its evolution with a Hamiltonian that involves just that system. In that case, the unitary evolution of vectors in the Hilbert space dictates a recipe for updating the density matrix $\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}_0\hat{U}(t,t_0)^\dagger$. More generally, a quantum system will get entangled with its environment, suffering

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an *incoherent evolution*. Under certain physically reasonable assumptions (cf. Appendix B), the equation that describes this dynamics is the *Lindblad master equation*:

$$\partial_t \hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathcal{L}_t(\hat{\rho}). \tag{2.26}$$

The linear *superoperator* $\mathcal{L}_t(\alpha\hat{\rho}_1 + \beta\hat{\rho}_2) = \alpha\mathcal{L}_t(\hat{\rho}_1) + \beta\mathcal{L}_t(\hat{\rho}_2)$, called the *Lindblad superoperator,* contains the information about the noise or the environment-induced decoherence. There are no general prescriptions to write down or even solve this kind of equation, but in a few cases the coupling is so weak and the environment so big that it instantaneously loses all memory about the system's dynamics. This is the so-called *Markovian limit*, in which $\mathcal{L}$ is independent of time and of the system's initial conditions. This limit provides a very simple and very accurate description of the loss of energy and of quantum coherence for many of the superconducting circuits that we will study – see, for instance, Section 5.5.2 or 6.1.5.

2.4 Measurements

The axioms of quantum mechanics prescribe the behavior of a quantum system under a complete measurement of any observable O, also known as *Von Neumann* or *projective* measurements. Each observable is associated to a different Hermitian operator $\hat{O}$. The eigenvalues that result from diagonalizing this operator o_n correspond to the possible measurement outcomes of the measurement. Let $\hat{P}_n = \sum_m |o_n, m\rangle\langle o_n, m|$ be the projector onto all quantum states for which the observable $\hat{O}$ has the value o_n. This projector is built using eigenstates of the observable $\hat{O}\,|o_n, m\rangle = o_n\,|o_n, m\rangle$, understanding that one measurement outcome may be given by many different quantum states, which differ in other generic quantum properties, here denoted as m. According to quantum mechanics, an *ideal projective measurement* of the observable $\hat{O}$ onto a state ρ will produce the outcome o_n with probability $p(o_n) = \mathrm{tr}(\hat{P}_n\hat{\rho})$. If the measurement is also *nondestructive*, the quantum state after the measurement will be projected onto a new density matrix:

$$\hat{\rho} \to \frac{1}{p(o_n)}\hat{P}_n\hat{\rho}\hat{P}_n. \tag{2.27}$$

We can highlight other properties and types of measurements. First, we may realize that the ideal measurement from (2.27) is an instance of a *quantum nondemolition* (QND) measurement, one which, if repeated twice on the same system, always produces the same expected value $\langle\hat{O}\rangle$. Ideal QND measurements are one of the targets for quantum computing setups. In those experiments, we need to determine the state of the superconducting qubit with certainty, through a projective

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measurement of the qubit's polarization σ^z, that leaves the qubit in a well-defined state $|0\rangle$ or $|1\rangle$. This way, we can use the outcome of the measurement as input to the following steps of quantum algorithms or to error-correction protocols.

Qubit measurements, such as used in those quantum computers, should ideally be *single shot*. This means that every time we run the experiment, we obtain a real value o_n associated to one measurement outcome, without errors. Note, however, that even if we obtain a meaningful value every time we measure, the estimation of $\langle \hat{O} \rangle$ or the probabilities $p(o_n)$ can still be *quantum limited*, and we may need to repeat the experiments many times to obtain such estimates with high accuracy. For instance, the unbiased estimator of the average using M experimental measurements $\bar{O}_{\text{est}} = \frac{1}{M} \sum_{i=1}^{M} o_{n_i}$ is itself a random variable with a standard deviation that approaches the quantum uncertainty:

$$\Delta \bar{O}_{\text{est}} = \frac{\Delta \hat{O}}{\sqrt{M}}. \tag{2.28}$$

Computing a good estimate means bringing this deviation down to zero, which we do by repeating the experiment again and again, until $\Delta \bar{O}_{\text{est}}$ lays below our desired tolerance.

In experiments, we rarely find direct projective measurements. More generally, experiments are designed so that we measure an auxiliary quantum object that has interacted with and extracted the information from the system we want to measure. The reason to operate this way is to reduce decoherence. If we connect an oscilloscope directly to a microwave resonator, the big classical object will quickly deteriorate the quantum state of the photons that are inside the cavity. It is therefore more convenient to create a setup such as the one in Figure 2.1, in which we perform a weak connection between the resonator and a superconducting waveguide that extracts only a tiny fraction of the photons, which are amplified and fed into a detector.

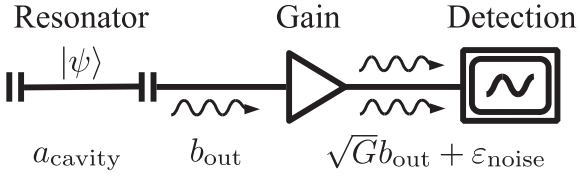

Figure 2.1 A resonator or cavity stores microwave photons whose state we wish to measure. Instead of connecting the measurement apparatus to the cavity, we make a weak connection between the cavity and an outgoing superconducting cable that, after passing through an amplifier, brings the signal to the measurement apparatus. The signal b_{out} is proportional to the cavity signal a_{cavity}, but contains vacuum noise. The measured signal, $Gb_{\text{out}} + \varepsilon$, is amplified with a gain $G > 1$, but contains additional noise ε from the amplifier.

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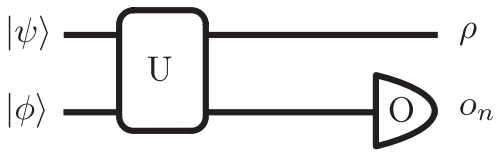

Figure 2.2 Quantum circuit for a generalized quantum measurement.

There are two consequences to using such a setup. First of all, we no longer have a *single-shot* measurement. Amplification introduces noise in the signal, meaning that our oscillator will retrieve a measurement $o_n + \varepsilon$ that is affected by random fluctuating photon noise ε. In absence of systematic errors, those errors average out $\langle\varepsilon\rangle = 0$, and we can still produce meaningful estimates $\bar{O} \simeq \frac{1}{M}\sum_m o_m$, but the presence of large noise prevents us from determining the quantum state of our system after the measurement – i.e., the measurement is no longer quantum limited.

The second consequence of indirect measurements is that we need a broader framework to understand both the measurement statistics and the state of the quantum system after a given measurement outcome. This framework is provided by *generalized quantum measurements*, depicted in Figure 2.2.

(1) The system ψ is put in contact with the auxiliary quantum object ϕ.
(2) Both systems interact through some unitary evolution, $\hat{U}$.
(3) We measure the auxiliary object using a projective measurement $\hat{O}$.

The generalized measurement or positive operator valued measurement (POVM) associates measurement outcomes o_m to operators $\hat{M}_n$ that are no longer projectors, but still satisfy some completeness relation:

$$\sum_n \hat{M}_n^\dagger \hat{M}_n = \mathbb{1}. \tag{2.29}$$

The POVM operators determine the statistics of the measurement outcomes $p(o_m) = \langle \hat{M}_n^\dagger \hat{M}_n \rangle$, and, if the measurement is nondestructive, also the post-measurement state of the quantum system:

$$|\psi\rangle \to \frac{\hat{M}_n |\psi\rangle\langle\psi| \hat{M}_n^\dagger}{\langle\psi|\hat{M}_n^\dagger \hat{M}_n|\psi\rangle}. \tag{2.30}$$

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