Solid state systems for quantum information, Correction 4

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Exercise 1 : The Cooper pair box (CPB) Hamiltonian and the transmon limit

From the second exercise sheet and from the lecture, you should have seen that the CPB Hamiltonian reads as:

$$\hat{H}_{CPB} = 4E_C\left(\hat{N} - n_g\right)^2 - E_J\cos\left(\hat{\delta}\right),$$

where $\hat{N}$ is the charge operator, $\hat{\delta}$ the phase operator, E_C the charging energy, E_J the Josephson energy and n_g an offset charge. $\hat{N}$ and $\hat{\delta}$ are conjugate operators, meaning that their commutator is $[\hat{\delta}, \hat{N}] = i$.

1. Show that $[\hat{N}, e^{i\hat{\delta}}] = e^{i\hat{\delta}}$.

2. Show that $e^{i\hat{\delta}}\,|m\rangle = |m+1\rangle$, where $|m\rangle$ is an eigenstate of the charge operator, i.e. $\hat{N}\,|m\rangle = m\,|m\rangle$.

3. Use the identities from 1. and 2. to show that the CPB Hamiltonian in the charge basis reads

$$\hat{H}_{CPB} = \sum_{m=-\infty}^{+\infty}\left[4E_C(m-n_g)^2\,|m\rangle\langle m| - \frac{1}{2}E_J(|m+1\rangle\langle m| + |m\rangle\langle m+1|)\right] \tag{1}$$

4. Numerically diagonalize the CPB Hamiltonian for various ratios of $E_J/E_C \in \{1, 5, 10, 30\}$ using qutip. For this purpose, truncate the Hilbert space at a suitable dimension corresponding to a maximum charge $|m_{max}|$.
   - Plot the eigenenergies $E_g(n_g)$, $E_e(n_g)$, $E_f(n_g)$, corresponding to the three lowest-lying eigenstates $|g\rangle$, $|e\rangle$, $|f\rangle$ of the CPB Hamiltonian as a function of n_g.
   - For $n_g = 0$, plot the probability $p_m = |\langle m|i\rangle|^2$ to be in charge state $|m\rangle$ for the two eigenstates $|i\rangle \in |g\rangle, |e\rangle$. Convince yourself that you have chosen a sufficiently large $|m_{max}|$ in your simulations.

5. From your numerical result from 4., extract the charge dispersion $\varepsilon = E_{ge}(n_g = 0) - E_{ge}(n_g = 1/2)$ of the $|g\rangle \leftrightarrow |e\rangle$ transition, where $E_{ge} = E_e - E_g$, for the four values of E_J/E_C, and plot them as function of E_J/E_C.

6. Show that the charge operator has the following representation in the phase eigenbasis $\{< |\delta\rangle\}$:

$$\langle\delta|\,\hat{N}\,|\Psi\rangle = i\frac{\partial}{\partial\delta}\Psi(\delta), \tag{2}$$

   where $\langle\delta|\Psi\rangle = \Psi(\delta)$ is the wave function in phase space. Make use of the charge basis representation of the phase eigenstates, $|\delta\rangle = \sum_{m=-\infty}^{+\infty} e^{im\delta}\,|m\rangle$.

Solution 1 :

1. We first expand the exponential in the commutator,

$$[\hat{N}, e^{i\hat{\delta}}] = \left[\hat{N}, \sum_{n=0}^{\infty} \frac{1}{n!}(i\hat{\delta})^n\right] = \sum_{n=0}^{\infty} \frac{1}{n!} i^n [\hat{N}, \hat{\delta}^n] = \sum_{n=1}^{\infty} \frac{1}{n!} i^n [\hat{N}, \hat{\delta}^n], \tag{3}$$

   with the sum starting at $n = 1$ since $[\hat{N}, \hat{\delta}^0] = 0$. We next calculate the commutator

$$\begin{aligned}[\hat{N}, \hat{\delta}^n] &= \hat{\delta}^{n-1}[\hat{N}, \hat{\delta}] + [\hat{N}, \hat{\delta}^{n-1}]\hat{\delta} \\ &= \hat{\delta}^{n-1} \underbrace{[\hat{N}, \hat{\delta}]}_{=-i} + \hat{\delta}^{n-2} \underbrace{[\hat{N}, \hat{\delta}]}_{=-i} \hat{\delta} + [\hat{N}, \hat{\delta}^{n-2}]\hat{\delta}^2 \\ &= \ldots \\ &= n(-i)\hat{\delta}^{n-1}.\end{aligned} \tag{4}$$

   Inserting this expression into Eq.3 yields

$$[\hat{N}, e^{i\hat{\delta}}] = \sum_{n=1}^{\infty} \frac{1}{n!} i^n n(-i)\hat{\delta}^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} i^{n-1}\hat{\delta}^{n-1} = \sum_{n'=0}^{\infty} \frac{1}{n'!} i^{n'}\hat{\delta}^{n'} = e^{i\hat{\delta}}, \tag{5}$$

   where we have substituted $n = n' + 1$ at the second-to-last equality.

2. Using the commutator from 2., we write

$$e^{i\hat{\delta}} |m\rangle = [\hat{N}, e^{i\hat{\delta}}] |m\rangle = \hat{N} e^{i\hat{\delta}} |m\rangle - e^{i\hat{\delta}} \hat{N} |m\rangle = \hat{N} e^{i\hat{\delta}} |m\rangle - m e^{i\hat{\delta}} |m\rangle \tag{6}$$

$$\Rightarrow \hat{N}\left(e^{i\hat{\delta}} |m\rangle\right) = (m+1)\left(e^{i\hat{\delta}} |m\rangle\right). \tag{7}$$

   From the last equation, we see that the state $e^{i\hat{\delta}} |m\rangle$ is an eigenstate of the charge operator $\hat{N}$ with eigenvalue $m + 1$, and therefore has to be identical to $|m+1\rangle$.

3. We write $\hat{H}_{CPB}$ in the charge basis by inserting twice the identity operator $\mathbb{1} = \sum_{m=-\infty}^{\infty} |m\rangle \langle m|$,

$$\hat{H}_{CPB} = \sum_{m'=-\infty}^{+\infty} |m'\rangle \langle m'| \left(4E_C(\hat{N} - n_g)^2 - E_J \cos\left(\hat{\delta}\right)\right) \sum_{m=-\infty}^{+\infty} |m\rangle \langle m| \tag{8}$$

$$= \sum_{m,m'=-\infty}^{+\infty} \Big[|m'\rangle \langle m'| 4E_C(\hat{N} - n_g)^2 |m\rangle \langle m| - |m'\rangle \langle m'| E_J \cos\hat{\delta} |m\rangle \langle m|\Big]. \tag{9}$$

   We evaluate the matrix elements

$$\langle m'| 4E_C(\hat{N} - n_g) |m'\rangle = 4E_C(m - n_g)^2 \langle m'|m\rangle = 4E_C(m - n_g)^2 \delta_{m',m} \tag{10}$$

$$\langle m'| E_J \cos\hat{\delta} |m\rangle = \frac{1}{2} E_J \langle m'| (e^{i\hat{\delta}} + e^{-i\hat{\delta}}) = \frac{1}{2} E_J \left(\langle m'| e^{i\hat{\delta}} |m\rangle + \langle m| e^{-i\hat{\delta}} |m'\rangle\right) \tag{11}$$

$$\overset{2.}{=} \frac{1}{2} E_J(\langle m'|m+1\rangle + \langle m'+1|m\rangle) = \frac{1}{2} E_J(\delta_{m',m+1} + \delta_{m'+1,m}), \tag{12}$$

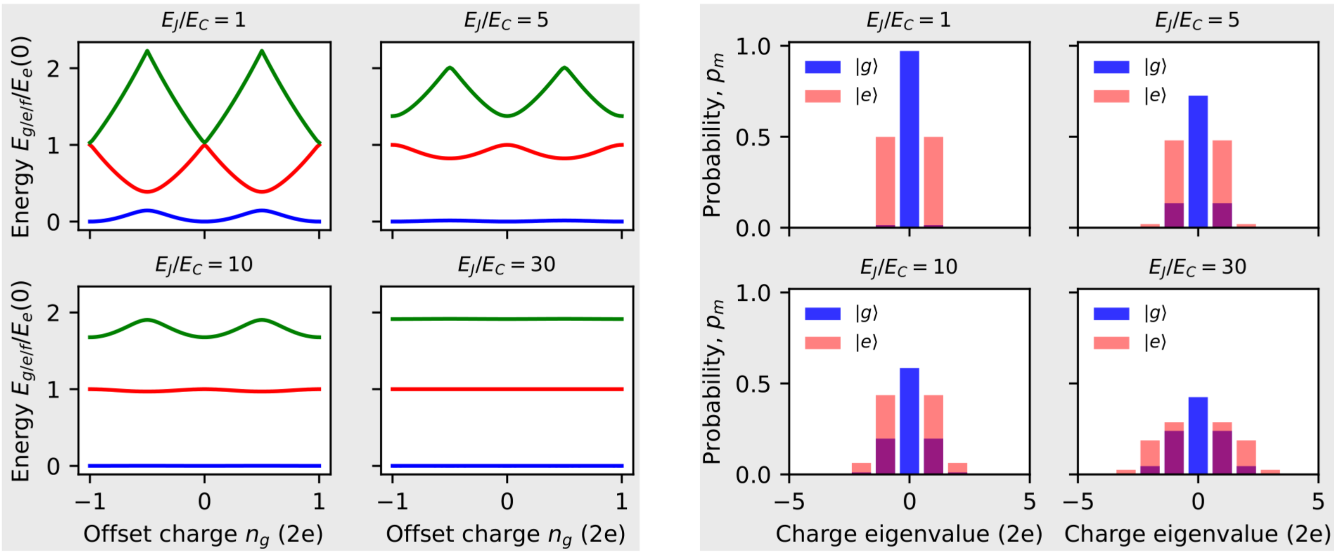

Figure 1: The first three energy levels of the CPB Hamiltonian as a function of n_g for four different values of E_J/E_C (left), and the probabilities p_m to be in charge state $|m\rangle$ for $n_g = 0$ for the ground and first excited state (right).

where $\delta_{i,j}$ is the Kronecker symbol ($\delta_{i,j} = 1$ if $i = j$ and 0 otherwise). Inserting the expressions for both matrix elements into Eq.8 and evaluating the Kronecker symbols yields

$$\hat{H}_{CPB} = \sum_{m=-\infty}^{+\infty} \left[|m\rangle\, 4E_C(m-n_g)^2 \langle m| - |m+1\rangle \frac{1}{2} E_J \langle m| - |m-1\rangle \frac{1}{2} E_J \langle m| \right] \tag{13}$$

$$= \sum_{m=-\infty}^{+\infty} \left[4E_C(m-n_g)^2 |m\rangle \langle m| - \frac{1}{2} E_J \left(|m+1\rangle \langle m| + |m\rangle \langle m+1| \right) \right], \tag{14}$$

where we have used

$$\sum_{m=-\infty}^{+\infty} |m-1\rangle \frac{1}{2} E_J \langle m| = \sum_{\tilde{m}=-\infty}^{+\infty} \frac{1}{2} E_J |\tilde{m}\rangle \langle \tilde{m}+1| \tag{15}$$

with $\tilde{m} = m - 1$ in the last term (which is possible because the summation runs from $-\infty$ to $+\infty$).

4. See solution jupyter notebook provided on Moodle. The calculated energy levels as a function of n_g are shown in the left plot of Fig.1. For $n_g = 0$, the probabilities p_m to be in charge state $|m\rangle$ are shown for the ground state and first excited state in the right plot of Fig.1.

5. See solution jupyter notebook provided on Moodle. Since the largest (smallest) $|g\rangle \leftrightarrow |e\rangle$ transition frequency is obtained for $n_g = 0$ ($n_g = 1/2$), we calculate the charge dispersion as

$$\varepsilon = (E_e(n_g = 0) - E_g(n_g = 0)) - (E_e(n_g = 1/2) - E_g(n_g = 1/2)). \tag{16}$$

The result is plotted in Fig.2 as a function of E_J/E_C. We observe that ε is efficiently suppressed as E_J/E_C increases (transmon regime). We evalutate $\langle\delta| \hat{N} |\psi\rangle$ by expanding $\langle\delta|$ in

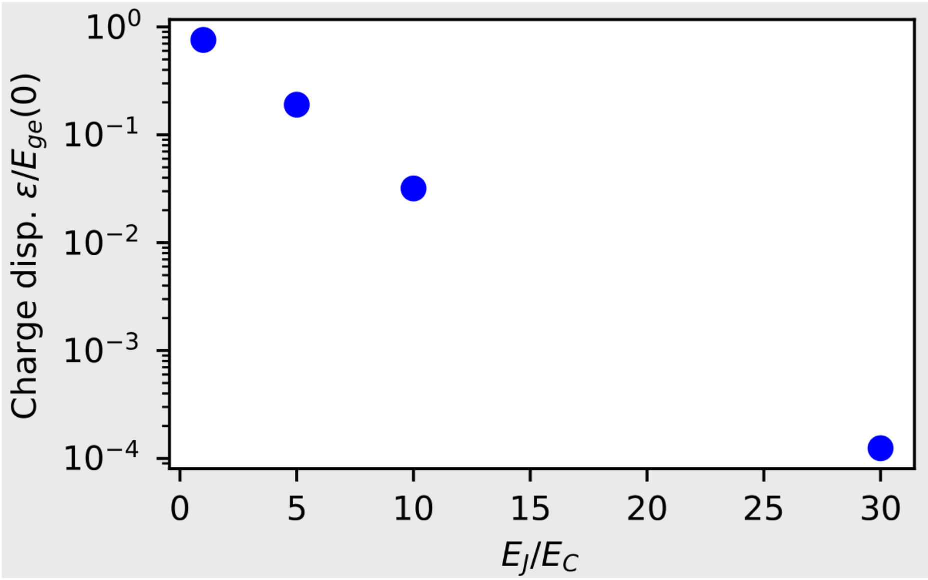

Figure 2: Charge dispersion ε normalized to the $|g\rangle \leftrightarrow |e\rangle$ transition frequency at $n_g = 0$ as a function of E_J/E_C.

the charge basis and applying the charge operator $\hat{N}$ from the right,

$$\langle\delta|\,\hat{N}\,|\psi\rangle = \sum_{m=-\infty}^{+\infty} e^{-im\delta}\,\langle m|\,\hat{N}\,|\psi\rangle = \sum_{m=-\infty}^{+\infty} m e^{-im\delta}\,\langle m|\psi\rangle\,. \tag{17}$$

On the other hand, we have

$$i\frac{\partial}{\partial\delta}\psi(\delta) = i\frac{\partial}{\partial\delta}\,\langle\delta|\psi\rangle = \sum_{m=-\infty}^{+\infty} i\frac{\partial}{\partial\delta}e^{-im\delta}\,\langle m|\psi\rangle = \sum_{m=-\infty}^{+\infty} \underbrace{i(-im)}_{=m}\,e^{-im\delta}\,\langle m|\psi\rangle\,, \tag{18}$$

which equals the right-hand side of Eq.17.