Solid state systems for quantum information, Correction 3

Assistants : franco.depalma@epfl.ch, filippo.ferrari@epfl.ch

1 Exercises

Exercise 1 : Capacitance of a coplanar plate capacitor

Superconducting circuits are based on inductive and capacitive elements patterned into a superconducting thin film, which resides on a dielectric substrate. Here, we estimate the capacitance and the dipole moment of a coplanar, parallel plate capacitor made of two rectangular pads, see Fig. 1. By combining such capacitors with inductive elements, one can e.g. build on-chip LC resonators.

1. Estimate the order of magnitude of the capacitance between the two pads of this element. Apply this to typical values $a = 300$ μm, $b = 400$ μm, $l = 100$ μm, film thickness $t = 150$ nm, and relative dielectric constant $\varepsilon_r \propto 10$. Assume for simplicity that the electric field lines extend mainly into the substrate and that the substrate thickness is large compared to the one of the capacitor.

2. Estimate the inductance required to obtain a lumped element resonator with a resonance frequency of $f \simeq 6$ GHz?

3. Estimate the dipole moment of this capacitor for a single Cooper pair located on one of the islands. The dipole operator in this example is given by $d = lQ$ with l in this case being the separation between the two pads and Q is the charge.

4. Calculate the charge zero point fluctuation, defined as $Q_{ZPF} = \sqrt{\frac{\hbar}{2Z}}$, where $Z = \sqrt{\frac{L}{C}}$. How does it compare to a single Cooper pair?

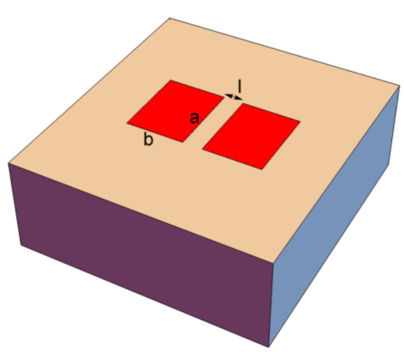

Figure 1: Sketch of a capacitor formed by two pads of a superconductor (red) on an insulating substrate (grey).

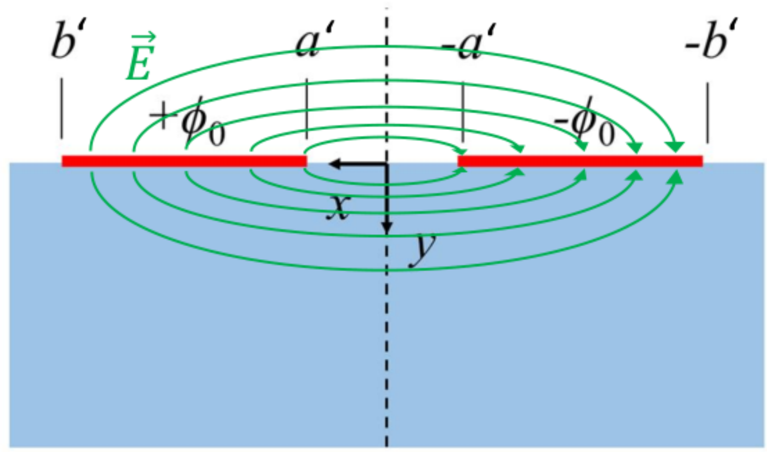

Figure 2: Cross section through the coplanar plate capacitor with the new variables a' and b' being introduced for the process of conformal mapping, see https://arxiv.org/pdf/1712.05079.pdf. The green field lines indicate a simplified distribution of the electric field $\vec{E}$.

Solution 1 :

1. Since the thickness of the superconductor is about three orders of magnitude smaller than all other dimensions, we can neglect the effect of the thickness for the following calculations and also assume that all electric field lines below the two plates are located within the dielectric substrate. We sketch the electric field distribution in Fig. 2 assuming a static electric potential with opposing sign $\pm\phi_0$ on both plates. All electric field lines start and end perpendicularly at the conducting plates of the capacitor and are distributed equally in both the vacuum and the substrate independently from the value of the dielectric constant ϵ_r.
   Since the total energy stored in volume V in the electric field $U = \int_V \epsilon_0 \epsilon_r |E|^2 dV = \frac{1}{2} C \Delta\phi_0^2$ is proportional to the dielectric constant ϵ_r, the contribution of the electric field lines through the vacuum to the total energy can be neglected.
   We can map the situation of the electric field onto two parallel plates with a separation of $d = l$ and an area of $A = ab$ and estimate the capacitance to be:
   $$C \approx \epsilon_0 \epsilon_r \frac{A}{d} = 8.8 \cdot 10^{-12} \cdot 10 \, \frac{\mathrm{F}}{\mathrm{m}} \cdot \frac{300 \cdot 10^{-6}\mathrm{m} \cdot 400 \cdot 10^{-6}\mathrm{m}}{100 \cdot 10^{-6}\mathrm{m}} \simeq 100 \cdot 10^{-15}\mathrm{F} = 100 \text{ fF}. \quad (1)$$
   By assuming a separation of $d = l$, we actually overestimate the total capacitance since the majority of the electric field lines in the coplanar structure do not directly connect to the plates at a separation of l.
   Additional Information: A more precise estimation of the capacitance can be obtained from the theory of conformal mapping, see https://arxiv.org/pdf/1712.05079.pdf. Based on the dimensions a' and b' in Fig. 1, we can introduce a geometric ratio $k = a'/b'$ and calculate the capacitance $C(k)$ according to the theory of conformal mapping. In Fig. 3, we compare the obtained capacitance C to our original estimate based on parallel plates with $d = l$ and also to the case of taking d as the distance of the center of both plates, $d = l + b$. We realize

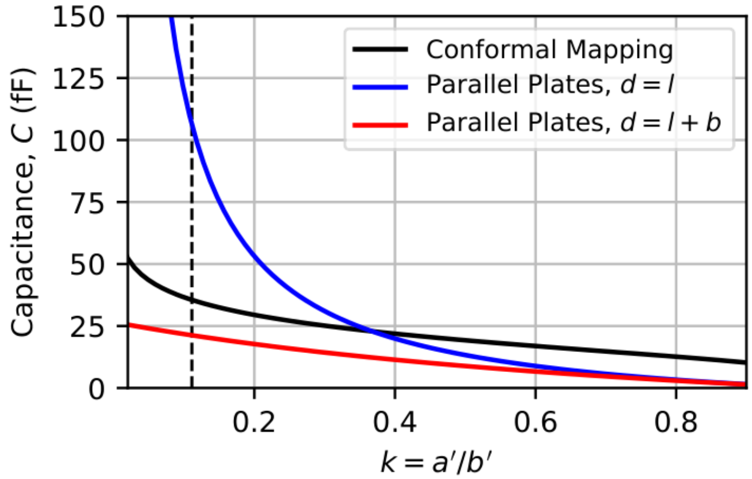

Figure 3: Total capacitance C as a function of the geometric ratio k for the different estimations methods. The dashed line indicates the geometric ratio resulting from the dimensions considered in this exercise.

that the initial approximation using $d = l(d = l + b)$ overestimates (underestimates) the total capacitance C for $k < 0.4$. For larger geometric ratios $k > 0.4$, the assumption of having two equivalent parallel plates breaks down, since the size of the plates is small compared to their separation.

2. The resonance frequency of an LC-resonator is given by $\omega = 1/\sqrt{LC}$. Thus we would need an inductance of

$$L = \frac{1}{\omega^2 C} = \frac{1}{(2\pi \cdot 6 \cdot 10^9 \frac{1}{\mathrm{s}}) \cdot 10^{-13} \frac{\mathrm{As}}{\mathrm{V}}} \approx 10^{-8}\mathrm{H} = 10\ \mathrm{nH} \tag{2}$$

3. The dipole is oriented from one pad to the next. We can estimate its magnitude as being roughly the separation distance, l, multiplied by the charge, which we take as $2e$ for a Cooper pair. We get $d \approx (2e)l = 2 \cdot 1.6 \cdot 10^{-19} \cdot 100 \cdot 10^{-6}\mathrm{C\ m} \simeq 3 \cdot 10^{-23}\mathrm{C\ m}$.

4. We can calculate the zero point fluctuations of the charge for our oscillator with $Q_{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2Z}}$ with $Z = \sqrt{L/C}$. One finds an impedance of about 300 Ω and $Q_{\mathrm{ZPF}} \simeq 4 \cdot 10^{-19}\mathrm{C}$, which is close to $2e$.

Exercise 2 : Input-output theory

We consider an LC-resonator which is capacitively coupled with a rate of κ_{ext} through capacitor C_c to a semi-infinite transmission line with a characteristic impedance of $Z_0 = 50\Omega$. The LC-resonator has an internal loss rate of κ_{int} and is depicted in Fig.4. We consider a classical coherent input field $\langle \hat{b}_{\text{int}}(t) \rangle = \beta_{in} e^{-i\omega t}$ traveling towards the LC-resonator, as well as an output field $\langle \hat{b}_{\text{out}}(t) \rangle$ being reflected from it. The equation of motion, as a function of time, for the system operator $\hat{a} = \hat{a}(t)$ reads

$$\frac{\partial \hat{a}}{\partial t} = \frac{i}{\hbar} \left[\hat{H}_{\text{sys}}, \hat{a} \right] - \frac{\kappa_{\text{int}} + \kappa_{\text{ext}}}{2} \hat{a} + \sqrt{\kappa_{\text{ext}}} \hat{b}_{in} \tag{3}$$

with $\hat{H}_{\text{sys}} = \hbar \omega_0 \hat{a}^\dagger \hat{a}$ being the system Hamiltonian of the LC-resonator. The boundary condition which relates the output to the input field is

$$\hat{b}_{\text{in}}(t) + \hat{b}_{\text{out}}(t) = \sqrt{\kappa_{\text{ext}}} \hat{a}(t) \tag{4}$$

1. Calculate the commutator $\left[\hat{H}_{\text{sys}}, \hat{a} \right]$ and use the Ansatz $\langle \hat{a}(t) \rangle = \alpha e^{-i\omega t}$ for the time dependence of the expectation value of the system operator and simplify the equation of motion. Express the amplitude α of the expectation value of the system operator as a function of the angular frequency ω and the amplitude of the input field β_{in}.

2. Solve for the frequency dependent reflection coefficient $S_{11}(\omega) = \frac{\langle \hat{b}_{\text{out}} \rangle}{\langle \hat{b}_{\text{in}} \rangle}$.

3. Plot the squared absolute value, and the real and imaginary parts of the reflection coefficient $S_{11}(\omega)$ for a resonator with parameters $\omega_0/2\pi = 6$ GHz and $\kappa_{\text{ext}}/2\pi = 5$ MHz. Use three different internal loss rates rates of $\kappa_{\text{int}}/2\pi \in \{0, 5, 100\}$ MHz. What changes when the internal loss rate is increased? Discuss all three cases.

   We now consider an LC-resonator which is capacitively coupled to two semi-infinite transmission lines, as shown Fig. 5.

4. Calculate the transmission coefficient $S_{21}(\omega) = \frac{\langle \hat{b}_{2,\text{out}} \rangle}{\langle \hat{b}_{1,\text{in}} \rangle}$. Show that the transmission coefficient $S_{21}(\omega = \omega_0) \rightarrow 1$ for a lossless LC-resonator.

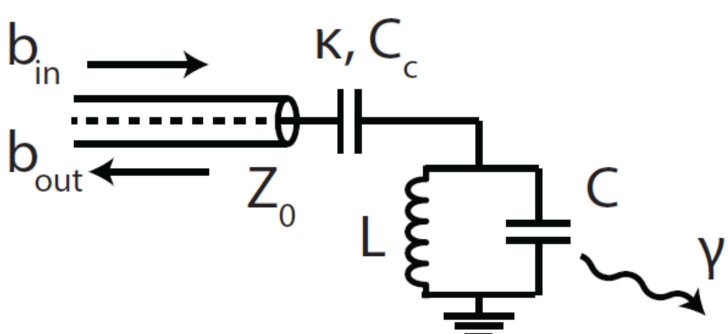

Figure 4: Electrical circuit of an LC-resonator capacitively coupled to a semi-infinite transmission line for a measurement of the reflection coefficient S_{11}.

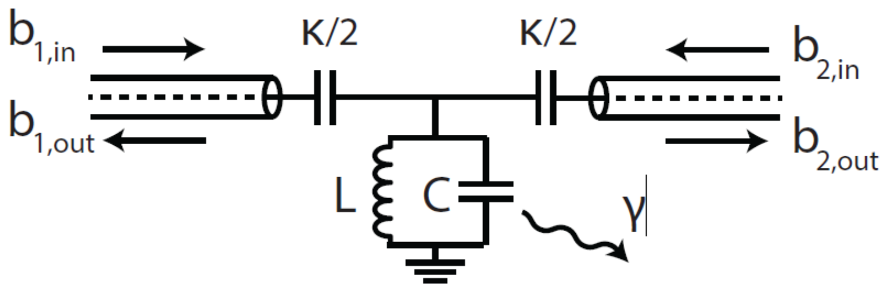

Figure 5: Electrical circuit of an LC-resonator capacitively coupled to two semi-infinite transmission lines for a measurement of the transmission coefficient S_{21}.

5. Plot the squared absolute value, and the real and imaginary parts of the transmission coefficient $S_{21}(\omega)$ for the same parameter values as in 3. In what way do the results differ from those for the reflection coefficient in 3.

Solution 2 :

1. We calculate the commutator by using $[a, a^\dagger a] = a$:

$$[a, H_{sys}] = \hbar\omega_0[a, a^\dagger a] = \hbar\omega_0 \left(a^\dagger[a, a] + [a, a^\dagger]a\right) = \hbar\omega_0 a. \tag{5}$$

We take the expectation value of equation 3 on both sides and obtain

$$\langle \dot{a}(t)\rangle = -i\omega_0 \langle a(t)\rangle - \frac{\kappa_{\text{ext}} + \kappa_{\text{int}}}{2} \langle a(t)\rangle + \sqrt{\kappa_{\text{ext}}} \langle b_{\text{in}}(t)\rangle \tag{6}$$

Using the Ansatz $\langle a(t)\rangle = \alpha \cdot e^{-i\omega t}$ and its time derivative $\langle \dot{a}(t)\rangle = -i\omega \langle a(t)\rangle$, we can rewrite the equation above as,

$$-i\omega\alpha \cdot e^{-i\omega t} = -i\omega_0\alpha \cdot e^{-i\omega t} - \frac{\kappa + \gamma}{2}\alpha \cdot e^{-i\omega t} + \sqrt{\kappa}\beta_{\text{in}} \cdot e^{-i\omega t}. \tag{7}$$

We can eliminate the harmonic time dependence $e^{-i\omega t}$ and simplify the equation of motion to

$$\alpha = \frac{\sqrt{\kappa}}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)}\beta_{\text{in}}. \tag{8}$$

2. We start by taking the expectation value of the boundary condition and obtain $\beta_{\text{in}} + \beta_{\text{out}} = \sqrt{\kappa}\alpha$. Based on equation 8 we now write the output field as a function of the input field,

$$\beta_{\text{out}} = \sqrt{\kappa}\alpha - \beta_{\text{in}} = \left(\frac{\kappa}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)} - 1\right)\beta_{\text{in}} = \left(\frac{\frac{\kappa-\gamma}{2} + i(\omega - \omega_0)}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)}\right)\beta_{\text{in}}. \tag{9}$$

Now we can directly calculate the reflection coefficient S_{11} as a ratio of the output and input field,

$$S_{11}(\omega) = \frac{\langle b_{\text{out}}\rangle}{\langle b_{\text{in}}\rangle} = \frac{\beta_{\text{out}}}{\beta_{\text{in}}} = \frac{\frac{\kappa-\gamma}{2} + i(\omega - \omega_0)}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)} = \frac{\kappa}{\kappa + \gamma} \cdot \frac{1}{\frac{1}{2} + i\frac{\omega_0-\omega}{\kappa+\gamma}} - 1 \tag{10}$$

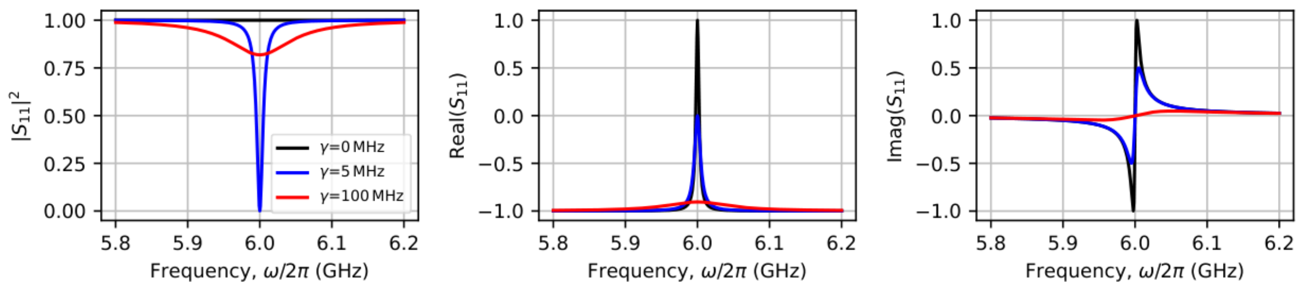

Figure 6: The squared absolute value, real part, and imaginary part of S_{11} for a resonator with $\omega_0/2\pi = 6$ GHz, $\kappa_{\text{ext}}/2\pi = 5$ MHz, and the three loss rates $\kappa_{\text{int}}/2\pi \in \{0, 5, 100\}$ MHz (black, blue, red).

3. The squared absolute value, real part and imaginary part of the reflection coefficient S_{11} are shown in Fig. 6. With a small internal loss rate $\gamma \ll \kappa$, we have an overcoupled resonator, where the decay is dominated by the coupling to the transmission line, and the squared absolute value of the reflection coefficient $|S_{11}|^2$ is independent of ω (see $\gamma = 0$). For similar rates $\gamma \approx \kappa$, we obtain a critical coupling for which the load impedance is matched to the source and we observe a reflection close to zero at ω_0. When the decay is dominated by the internal loss rate $\gamma \gg \kappa$, the absolute value of the reflection coefficient at ω_0 is between the overcoupled and the critically coupled case.

4. We begin formulating the equation of motion for the case of two coupled ports with $\kappa_1 = \kappa_2 = \kappa/2$ and loss rate γ,

$$\dot{a} = -\frac{i}{\hbar}[a, H_{\text{sys}}] - \frac{\kappa_1}{2}a + \sqrt{\kappa_1}b_{1,\text{in}} - \frac{\kappa_2}{2}a + \sqrt{\kappa_1}b_{2,\text{in}} - \frac{\gamma}{2}a \tag{11}$$

$$\dot{a} = -\frac{i}{\hbar}[a, H_{\text{sys}}] - \frac{\kappa+\gamma}{2}a + \sqrt{\frac{\kappa}{2}}(b_{1,\text{in}} + b_{2,\text{in}}). \tag{12}$$

We take the expectation value of equation 12 on both sides and obtain

$$\langle \dot{a}(t)\rangle = -\frac{i}{\hbar}\hbar\omega_0 \langle a(t)\rangle - \frac{\kappa+\gamma}{2}\langle a(t)\rangle + \sqrt{\frac{\kappa}{2}}(\langle b_{1,\text{in}}(t)\rangle + \langle b_{2,\text{in}}(t)\rangle). \tag{13}$$

Using the Ansatz, $\langle a(t)\rangle = \alpha \cdot e^{-i\omega t}$ and following the same procedure as in question 1., we obtain

$$\alpha = \frac{\sqrt{\frac{\kappa}{2}}}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)}(\beta_{1,\text{in}} + \beta_{2,\text{in}}). \tag{14}$$

Now, we take the expectation value of the boundary condition for the second port yielding

$$\beta_{2,\text{out}} = \sqrt{\frac{\kappa}{2}}\alpha - \beta_{2,\text{in}}. \tag{15}$$

Finally, we solve for the transmission coefficient $S_{21}(\omega)$ under the constraint that $\beta_{2,\text{in}} = 0$

$$S_{21}(\omega) = \frac{\langle b_{2,\text{out}}\rangle}{\langle b_{1,\text{in}}\rangle} = \frac{\beta_{2,\text{out}}}{\beta_{1,\text{in}}} = \frac{\kappa/2}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)}. \tag{16}$$

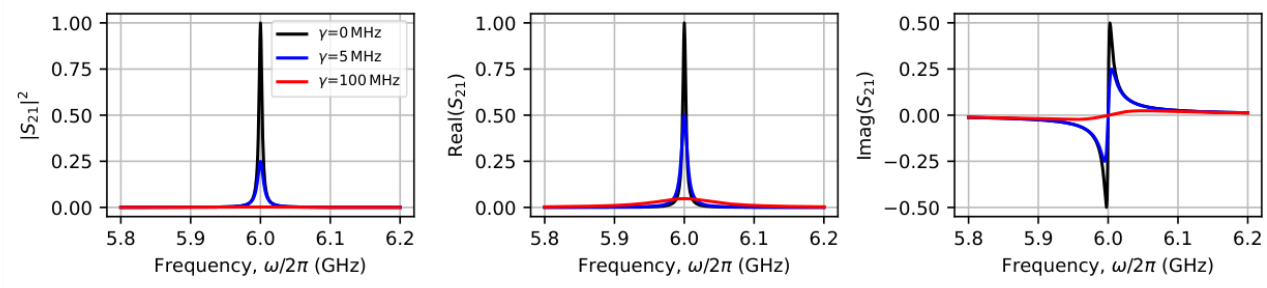

Figure 7: The squared absolute value, real part, and imaginary part of S_{21} for a resonator with $\omega_0/2\pi = 6$ GHz, $\kappa_{\mathrm{ext}}/2\pi = 5$ MHz, and the three loss rates $\kappa_{\mathrm{int}}/2\pi \in \{0, 5, 100\}$ MHz (black, blue, red).

We can see that the transmission coefficient S_{21} approaches one for $\gamma = 0$ and $\omega = \omega_0$

$$S_{21} = \frac{\kappa/2}{\frac{\kappa+\gamma}{2} - i(\omega - \omega_0)} \overset{\omega\to\omega_0}{=} \frac{\kappa/2}{\frac{\kappa+\gamma}{2}} \overset{\gamma\to 0}{=} 1. \tag{17}$$

5. The squared absolute value, real part and imaginary part of the transmission coefficient S_{21} are shown in Fig. 7. Compared to the measurement of the resonator in reflection, the squared absolute value of the transmission coefficient $|S_{21}|^2$ is always minimal unless the resonator is driven close to ω_0.

Exercise 3 : Coupled-cavity Arrays

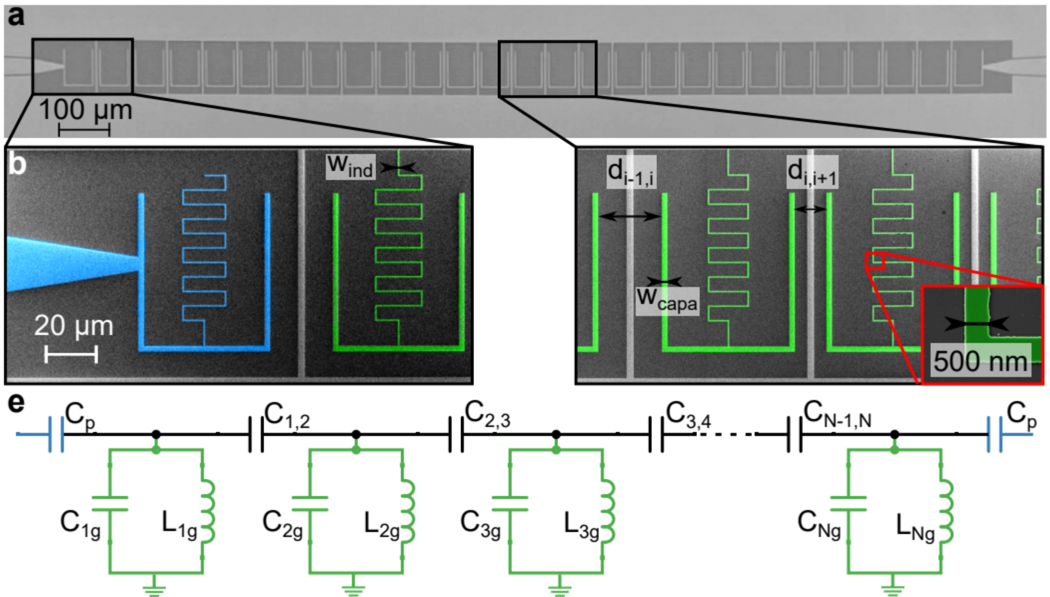

Figure 8: Coupled cavity array made of a chain of coupled LC resonators. The figure is taken from https://arxiv.org/abs/2403.18150.

In this exercise we study the transmission coefficients of coupled cavity arrays. We consider the system described in Fig. 8, which is chain of coupled L LC resonators.

1. Quantize the circuit and write down the equivalent Hamiltonian in terms of bosonic creation and annihilation operators $\hat{a}_\ell$ and $\hat{a}_\ell^\dagger$.

2. Assume periodic boundary conditions and diagonalize the Hamiltonian by passing in Fourier space by means of the momentun representation of the creation and annihilation operators

$$\hat{a}_\ell = \frac{1}{\sqrt{L}} \sum_k \hat{a}_k e^{ikx_\ell}, \qquad \hat{a}_\ell^\dagger = \frac{1}{\sqrt{L}} \sum_k \hat{a}_k^\dagger e^{-ikx_\ell}. \tag{18}$$

3. Suppose that now all the resonators have an intrinsic decay rate equal to κ_{int}, and that the first and the last resonators, due to the coupling with external feedlines, have an additional external decay rate equal to κ_{ext}. Write down the input-output equations for the intracavity field $\langle \hat{a}_\ell \rangle$.

4. Re-write the input-output relations in the compact form

$$\frac{\partial}{\partial t} \langle \hat{\vec{a}} \rangle = i\mathcal{M} \langle \hat{\vec{a}} \rangle, \tag{19}$$

where $\mathcal{M}$ is a $L \times L$ non-Hermitian matrix. Show that it is possible to diagonalize $\mathcal{M}$ with a matrix transformation, obtaining eigenmodes and eigenfrequencies of the system. Numerically diagonalize $\mathcal{M}$ for chains with $L = 10$ and $L = 100$ resonators. Discuss the physical interpretation of the system's eigenfrequencies, in light of what you found in point 2.

5. We now want to explore the response of the system to the injection of external photons, that we model with an external classical drive on the first cavity, $\langle \hat{b}_{\rm in}(t) \rangle = \beta_{\rm in} e^{i\omega t}$. Modify the input-output equations to account for the presence of the external drive. Derive the transmission coefficient $S_{1,L}(\omega)$ in terms of the eigenvectors and eigenvalues of $\mathcal{M}$. Plot $S_{1,L}(\omega)$ as a function of the drive frequency ω for chains of $L = 10$ and $L = 50$ resonators. Use the following parameters: $\omega_0/2\pi = 6$ GHz, $\kappa_{\rm int}/2\pi = 100$ kHz, $\kappa_{\rm ext}/2\pi = 10$ MHz, $g/2\pi = 200$ MHz.

Solution 3 :

1. The Hamiltonian of the coupled-cavity array reads ($\hbar = 1$)

$$\hat{H} = \sum_{\ell=1}^{L} \omega_0 \hat{a}_\ell^\dagger \hat{a}_\ell - g \sum_{\ell=1}^{L-1} \left(\hat{a}_{\ell+1}^\dagger \hat{a}_\ell + \hat{a}_\ell^\dagger \hat{a}_{\ell+1} \right), \tag{20}$$

which coincides with a tight-binding chain.

2. We now diagonalize the above Hamiltonian assuming periodic boundary conditions. Since $\hat{H}$ (once periodic boundary conditions have been assumed) is space-translational invariant, a Fourier transformation diagonalizes it. More specifically, we consider the Fourier transform of the creation and annihilation operators as

$$\hat{a}_\ell = \frac{1}{\sqrt{L}} \sum_k \hat{a}_k e^{ikx_\ell}, \qquad \hat{a}_\ell^\dagger = \frac{1}{\sqrt{L}} \sum_k \hat{a}_k^\dagger e^{-ikx_\ell}. \tag{21}$$

By plugging the above identities in the Hamiltonian we write down

$$\begin{aligned} \hat{H} &= \frac{\omega_0}{L} \sum_{\ell=1}^{L} \sum_{k,k'} \hat{a}_k^\dagger \hat{a}_{k'} e^{-ix_\ell(k-k')} - \frac{g}{L} \sum_{\ell=1}^{L-1} \sum_{k,k'} \left[\hat{a}_k^\dagger \hat{a}_{k'} e^{-ix_\ell(k-k')} e^{-ik} + \hat{a}_{k'}^\dagger \hat{a}_k e^{ix_\ell(k-k')} e^{ik'} \right] \\ &= \sum_k \omega_0 \hat{a}_k^\dagger \hat{a}_k - g \sum_k \left(\hat{a}_k^\dagger \hat{a}_k e^{-ik} + \hat{a}_k^\dagger \hat{a}_k e^{ik} \right) = \sum_k \varepsilon_k \hat{a}_k^\dagger \hat{a}_k, \end{aligned} \tag{22}$$

where $\varepsilon_k = \omega_0 - 2g\cos(k)$ is the tight-binding dispersion relation. To pass from the first to the second line we used the indentity $\frac{1}{L}\sum_{\ell=1}^{L} e^{ix_\ell(k-k')} = \delta(k-k')$. Notice that, in Fourier space, the frequencies of the coupled-cavity array are contained in the interval $[\omega_0 - 2g, \omega_0 + 2g]$.

3. The input-output equations for the bare coupled-cavity array read

$$\frac{\partial \hat{a}_\ell}{\partial t} = i[\hat{H}, \hat{a}_\ell] - \frac{\kappa_{\rm tot}}{2} \hat{a}_\ell, \tag{23}$$

where $\kappa_{\rm tot} = \kappa_{\rm int} + \kappa_{\rm ext}$ if $\ell = 1, L$ and $\kappa_{\rm tot} = \kappa_{\rm int}$ otherwise. If we compute the commutator we obtain the following equation for the field's coherence $\langle \hat{a}_\ell \rangle$

$$\frac{\partial \langle \hat{a}_\ell \rangle}{\partial t} = -\left(i\omega_0 + \frac{\kappa_{\rm tot}}{2} \right) \langle \hat{a}_\ell \rangle + ig \left[\langle \hat{a}_{\ell+1} \rangle + \langle \hat{a}_{\ell-1} \rangle \right]. \tag{24}$$

4. We now introduce the vector of the field coherences $\langle\hat{\vec{a}}\rangle = [\langle\hat{a}_1\rangle, ..., \langle\hat{a}_L\rangle]^{\mathrm{T}}$. The above input-output equation straightforwardly becomes

$$\frac{\partial}{\partial t}\langle\hat{\vec{a}}\rangle = i\mathcal{M}\langle\hat{\vec{a}}\rangle, \tag{25}$$

where

$$\mathcal{M} = \begin{pmatrix} \omega_0 - \frac{i}{2}(\kappa_{\rm int}+\kappa_{\rm ext}) & g & 0 & \cdots & 0 \\ g & \omega_0 - \frac{i}{2}\kappa_{\rm int} & g & \cdots & 0 \\ 0 & g & \omega_0 - \frac{i}{2}\kappa_{\rm int} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_0 - \frac{i}{2}(\kappa_{\rm int}+\kappa_{\rm ext}). \end{pmatrix} \tag{26}$$

Notice that $\mathcal{M}$ is non-Hermitian. Now we introduce the transformation $\mathcal{U}$ which diagonalizes $\mathcal{M}$, $\mathcal{U}\mathcal{M}\mathcal{U}^{-1} = \mathcal{D}$ (since $\mathcal{M}$ is non-Hermitian, $\mathcal{U}$ is not necessarily unitary), being $\mathcal{D}$ the diagonal matrix containing the (complex) dressed frequencies $(\widetilde{\omega}_1, ..., \widetilde{\omega}_L)$. Then we have that

$$\mathcal{U}\frac{\partial}{\partial t}\langle\hat{\vec{a}}\rangle = i\mathcal{U}\mathcal{M}\langle\hat{\vec{a}}\rangle = i(\mathcal{U}\mathcal{M}\mathcal{U}^{-1})\mathcal{U}\langle\hat{\vec{a}}\rangle = i\mathcal{D}\mathcal{U}\langle\hat{\vec{a}}\rangle. \tag{27}$$

We can then formally write the solution of the input-output equations as

$$\widetilde{\langle\hat{a}_\ell\rangle}(t) = e^{i\widetilde{\omega}_\ell t}\widetilde{\langle\hat{a}_\ell\rangle}(0). \tag{28}$$

The eigenfrequencies $\widetilde{\omega}_\ell$ are complex numbers with a real part which is the true frequency of the ℓ-th mode, and a purely imaginary part which represents a linewidth. In particular, $\widetilde{\omega}_\ell \neq \omega_0$ because of modes' hibridization, and you can check that $\widetilde{\omega}_\ell \in [\omega_0 - 2g, \omega_0 + 2g]$, while the imaginary part coincides with $\kappa_{\rm tot}/2$.

5. Now we derive the transmission coefficient $S_{1,L}(\omega)$ (which is a quantity that can be easily measured in the lab) for the coupled cavity array. The formal definition reads

$$S_{1,L}(\omega) = \frac{\langle\hat{b}_{L,\rm out}\rangle}{\langle\hat{b}_{1,\rm in}\rangle}. \tag{29}$$

Let's compute $\langle\hat{b}_{L,\rm out}\rangle$ using input-output theory. The input-output equations reads

$$\frac{\partial\hat{a}_\ell}{\partial t} = i[\hat{H}, \hat{a}_\ell] - \frac{\kappa_{\rm tot}}{2}\hat{a}_\ell + \sqrt{\kappa_{\rm ext}}\hat{b}_{\rm in}\delta_{1,\ell}. \tag{30}$$

In matrix form, once expectation values of operators are taken, they read

$$\frac{\partial}{\partial t}\langle\hat{\vec{a}}\rangle = i\mathcal{M}\langle\hat{\vec{a}}\rangle + \sqrt{\kappa_{\rm ext}}\langle\hat{\vec{b}}_{\rm in}\rangle. \tag{31}$$

We now propose the coherent ansatz for the expectation values of the bosonic fields, namely $\langle\hat{\vec{a}}\rangle = e^{i\omega t}\vec{\alpha}$, where ω is the drive frequency of the $\hat{b}_{\rm in}$ mode. By injecting the ansatz in the above equation we obtain

$$(\mathbb{1}\omega - \mathcal{M})\vec{\alpha} = -i\sqrt{\kappa_{\rm ext}}\vec{\beta}_{\rm in}. \tag{32}$$

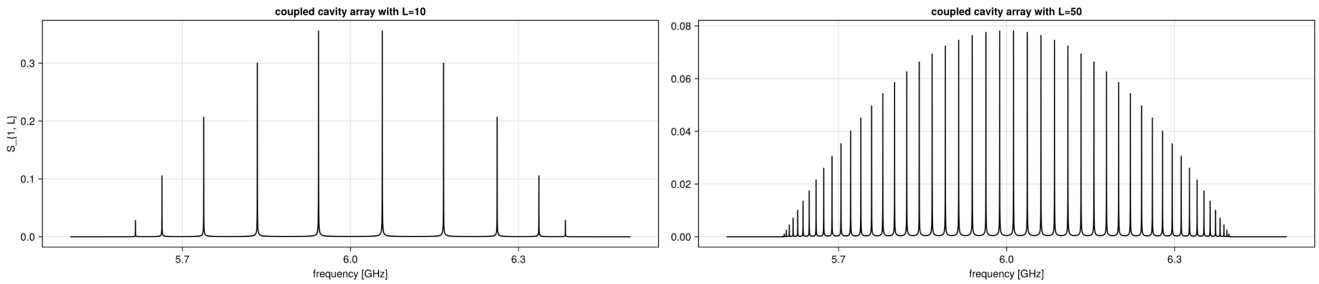

Figure 9: Transmission spectrum of the coupled cavity array $S_{1,L}$ as a function of the drive frequency and for the parameters $\omega_0/2\pi = 6$ GHz, $\kappa_{\text{int}}/2\pi = 100$ kHz, $\kappa_{\text{ext}}/2\pi = 10$ MHz, $g/2\pi = 200$ MHz.

To find $\vec{\alpha}$, we need to invert the matrix $(\mathbb{1}\omega - \mathcal{M})$. This can be done by expanding $\mathcal{M}$ in its eigenmodes and eigenfrequencies. We have that

$$\mathcal{M} = \sum_k E_k \left|\psi_k\right\rangle\!\left\langle\psi_k\right|, \qquad \text{with} \qquad E_k = \widetilde{\omega}_k - \frac{i\kappa_{\text{tot}}}{2}, \tag{33}$$

and $|\psi_k\rangle$ are the eigenmodes of the system. At this point

$$(\mathbb{1}\omega - \mathcal{M}) = \sum_k \left[(\omega - \widetilde{\omega}_k) + \frac{i\kappa_{\text{tot}}}{2}\right] \left|\psi_k\right\rangle\!\left\langle\psi_k\right|, \tag{34}$$

and since the eigenmodes span a L-dimensional orthonormal basis we can write

$$(\mathbb{1}\omega - \mathcal{M})^{-1} = \sum_k \frac{\left|\psi_k\right\rangle\!\left\langle\psi_k\right|}{(\omega - \widetilde{\omega}_k) + i\kappa_{\text{tot}}/2} \tag{35}$$

And finally we have

$$\vec{\alpha} = \sum_k \frac{-i\sqrt{\kappa_{\text{ext}}}\left|\psi_k\right\rangle\!\left\langle\psi_k\right|}{(\omega - \widetilde{\omega}_k) + i\kappa_{\text{tot}}/2}\vec{\beta}_{\text{in}}. \tag{36}$$

To find the transmission coefficient, we project $\vec{\alpha}$ on the computational states $|1\rangle$ and $|L\rangle$, which are just $|1\rangle = (1, 0, 0, ..., 0)^{\text{T}}$, and $|L\rangle = (0, ..., 0, 0, 1)^{\text{T}}$. We finally use the input-output relation $\vec{\beta}_{L,\text{out}} = \vec{\beta}_{L,\text{in}} - \sqrt{\kappa_{\text{ext}}}\vec{\alpha}$ (where $\vec{\beta}_{L,\text{in}} = 0$) and the transmission coefficient reads

$$S_{1,L}(\omega) = \sum_k \frac{i\kappa_{\text{ext}}\langle 1\,|\psi_k\rangle\,\langle\psi_k\,|L\rangle}{(\omega - \widetilde{\omega}_k) + i\kappa_{\text{tot}}/2} = \sum_k \frac{i\kappa_{\text{ext}} M_{1,L}}{(\omega - \widetilde{\omega}_k) + i\kappa_{\text{tot}}/2}. \tag{37}$$

If we take $|S_{1,L}|$ we get a sum of peaks centered at the frequencies of the eigenmodes $\widetilde{\omega}_k$ with a linewidth given by κ_{tot}, and an amplitude controlled by the external dissipation rate κ_{ext} and the matrix elements of the eigenmodes $M_{1,L}$.

Exercise 4 : Superconducting loops

In the superconducting state of a metal, the electrons close to the Fermi energy pair up with opposite spin and momentum to form Cooper pairs. All the Cooper pairs will have the same energy and they realize a macroscopic quantum state across the entire piece of the superconductor. Because this superconducting condensate is a coherent quantum state, it can be described by a wavefunction, which takes the form of

$$\psi(\mathbf{r}) = \sqrt{n_s} e^{i\varphi(\mathbf{r})} \tag{38}$$

where $\boldsymbol{r}$ is the position inside the superconductor, n_s is the density of the superconducting Cooper pairs and $\varphi(\boldsymbol{r})$ is the phase of the wavefunction. While inside the superconductor the amplitude of the wavefunction $\sqrt{n_s}$ is constant, the phase $\varphi(\boldsymbol{r})$ can change as a function of position. In this problem, we investigate how the phase behaves in a cylindrical superconductor in magnetic field.

1. In class, we saw that the change in phase with a current flowing in a superconductor from a point X to Y was defined as:

$$\varphi_{XY} = \frac{1}{\hbar} \int_X^Y p(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{39}$$

where $p = 2m\boldsymbol{v} + 2e\boldsymbol{A}$ in a magnetic field $\boldsymbol{B}$, with $\boldsymbol{A}$ being the vector potential respecting the relation $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. We consider a thin cylindrical superconductor with radius R and long axis oriented along the z direction (see Fig. 10). The external magnetic field also points along the z axis, and has a magnitude of B_0 such that $\boldsymbol{B} = (0, 0, B_0)$ in the Cartesian $x - y - z$ coordinate system. Importantly, the choice of the vector potential $\boldsymbol{A}$ is not unique; as long as it satisfies that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the vector potential describes correctly the system. Consequently, the phase of the superconductor is also not unique, which is not surprising because the phase of a wavefunction is not an observable quantity. In general, the vector potential can be changed by a gauge transformation such that

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi, \tag{40}$$

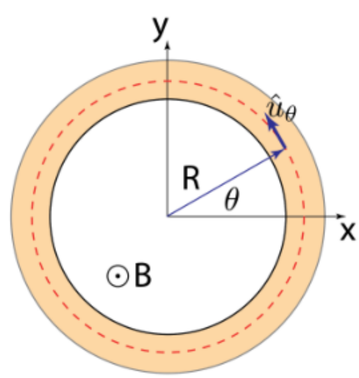

Figure 10: Top view of the cylindrical superconductor.

where χ is a scalar function. Here, we consider the Coulomb gauge, i.e. $\nabla\chi = 0$, so that $\boldsymbol{A} = (-yB_0/2, xB_0/2, 0)$.

Knowing that the current density $\boldsymbol{J}$ is given by $\boldsymbol{J} = 2en_s\boldsymbol{v}$, write the phase difference as a function of $\boldsymbol{J}$ and $\boldsymbol{A}$. Then show that the magnetic flux through the loop is quantized, by integrating over a closed loop and using the fact that the superconducting phase is periodic.

2. Now, let us consider a superconducting loop that contains two Josephson junctions (JJs), as depicted in Fig. 11, which is called a superconducting quantum interference device (SQUID). The current through a JJ can be described by the first Josephson equation:

$$I(\delta) = I_0 \sin(\delta), \tag{41}$$

where I_0 is the critical current of the junction, which depends on its geometry, and δ is the phase difference between the superconductors before and after the junction.

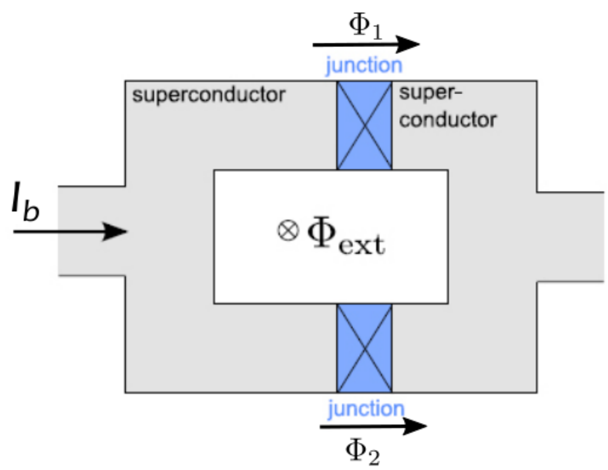

Figure 11: SQUID loop.

(a) Assuming that both JJs have the same critical current, what is the total current through the SQUID as a function of the phase differences δ_1 and δ_2 across JJs 1 and 2, respectively?

(b) Now we want to investigate the phase across the loop. In question 1., the gauge choice for $\boldsymbol{A}$ did not matter, since the contribution of χ vanishes when integrating over a closed loop. Since we now have two JJs in the loop, we cannot take a closed loop integral through the superconductor anymore. Therefore, we have to redefine the phase difference δ such that it becomes gauge invariant:

$$\delta = \varphi_Y - \varphi_X - \frac{2\pi}{\Phi_0}\int_X^Y \boldsymbol{A}(\boldsymbol{r})d\boldsymbol{r}, \tag{42}$$

where φ_X and φ_Y are the superconducting phases before and after the JJ, respectively. Calculate the phase change across the SQUID by splitting the integral into four parts, two parts through the superconductor and one part across each JJ.

(c) Using the result from a) and b), write the current total current through the SQUID as a function of the magnetic flux through the loop. Discuss how this result can be used to tune SQUIDs in a real device.

Solution 4 :

1. We can solve $\boldsymbol{J} = 2en_s\boldsymbol{v}$ for $\boldsymbol{J}$ and insert it in $p = 2m\boldsymbol{v} + 2e\boldsymbol{A}$ to get

$$\Delta\varphi_{XY} = \frac{m}{\hbar e n_s}\int_X^Y \boldsymbol{J}(\boldsymbol{r})d\boldsymbol{r} + \frac{2\pi}{\Phi_0}\int_X^Y \boldsymbol{A}(\boldsymbol{r})d\boldsymbol{r},$$

where we used the magnetic flux quantum $\Phi_0 = h/2e$.

The first integral vanishes, because the current in the superconductor is zero, assuming it is thicker than the London penetration depth. This can be explained by using Ampère's law $\boldsymbol{J} = \mu_0\nabla\times\boldsymbol{B}$ and the Meissner effect, which states that external magnetic fields are repelled from a superconductor. This is achieved by circular currents induced in the surface of the superconductor, screening the magnetic field, which leads to an external decay until magnetic field vanishes completely. Thus, the first integral vanishes only, when we choose a contour inside the superconductor.

The second integral can most easily be solved by changing to polar coordinates with $\hat{\boldsymbol{u}}_0 = (-\sin(\theta), \cos(\theta), 0)$, $x = R\cos(\theta)$ and $y = R\sin(\theta)$:

$$\Delta\varphi_{XY} = \frac{2\pi}{\Phi_0}\oint_0^{2\pi} \boldsymbol{A}\cdot\hat{\boldsymbol{u}}_0 R d\theta = \frac{2\pi}{\Phi_0}\oint_0^{2\pi}\frac{B_0}{2}R^2 d\theta = 2\pi\frac{\Phi}{\Phi_0} = 2\pi n,$$

where we have used $\Phi = B_0\pi R^2$ for the magnetic flux through the loop and the fact that the superconducting phase is 2π-periodic. From this we can see that the magnetic flux through a superconducting loop is always equal to a multiple of the magnetic flux quantum Φ_0. This is ensured by superconducting currents in the surface of the superconductor.

2. (a) We can just add up Eq.41 for two junctions with a critical current of $I_0/2$ and get

$$I_{\text{tot}} = I_1 + I_2 = \frac{I_0}{2}(\sin(\delta_1) + \sin(\delta_2)) = I_0\left(\cos\left(\frac{\delta_1-\delta_2}{2}\right)\sin\left(\frac{\delta_1+\delta_2}{2}\right)\right),$$

using trigonometric identities.

(b) To get the phase change through a SQUID, we use the same strategy as in question 1, but this time we have to split the loop integral into four parts:

$$\oint_C \nabla\varphi d\boldsymbol{r} = (\varphi_b - \varphi_a) + (\varphi_c - \varphi_b) + (\varphi_d - \varphi_c) + (\varphi_a - \varphi_d)$$

For the phase difference across the superconductor, we get

$$\varphi_c - \varphi_b = \frac{2\pi}{\Phi_0}\int_b^c \boldsymbol{A}d\boldsymbol{r} + \frac{m}{\hbar e n_s}\int_b^c \boldsymbol{J}d\boldsymbol{r}$$
$$\varphi_a - \varphi_d = \frac{2\pi}{\Phi_0}\int_d^a \boldsymbol{A}d\boldsymbol{r} + \frac{m}{\hbar e n_s}\int_d^a \boldsymbol{J}d\boldsymbol{r}.$$

For the phase difference across the JJs, we use the definition of the gauge invariant phase difference given in Eq.42

$$\varphi_b - \varphi_a = -\delta_1 + \frac{2\pi}{\Phi_0}\int_a^b \boldsymbol{A} d\boldsymbol{r}$$
$$\varphi_d - \varphi_c = \delta_2 + \frac{2\pi}{\Phi_0}\int_c^d \boldsymbol{A} d\boldsymbol{r}.$$

Summing up all the segments, we get

$$\oint_C \nabla\varphi d\boldsymbol{r} = 2\pi n = \delta_2 - \delta_1 + \frac{2\pi}{\Phi_0}\oint_C \boldsymbol{A} d\boldsymbol{r} + \frac{m}{\hbar e n_s}\oint_{C'} \boldsymbol{J} d\boldsymbol{r},$$

where C' is the incomplete contour excluding the path through the JJs.

Strictly speaking, this time the current density is not always zero along C', but if we assume that the JJs are sufficiently small (much smaller than the London penetration depth of the superconductor), the external magnetic field is sufficiently suppressed by the two sides of the superconductor enclosing the JJs, so current density is indeed zero and the integral vanishes again.

Thus, we now have

$$\delta_2 - \delta_1 = 2\pi\frac{\Phi}{\Phi_0} + 2\pi n$$

(c) Inserting the result from b) into the result from a) for the total current results in

$$I_{tot} = I_0\left(\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sin\left(\delta_1 + \frac{\pi\Phi}{\Phi_0}\right)\right).$$

Using the definition of the inductance $V(t) = L\frac{\partial I}{\partial t} = L\frac{\partial I}{\partial \delta}\frac{\partial \delta}{\partial t}$ and the second Josephson equation $\frac{\partial \delta}{\partial t} = \frac{V}{\Phi_0}$ we can derive the inductance of the SQUID:

$$L = \frac{\Phi_0}{2\pi I_0 \cos\left(\frac{\pi\Phi}{\Phi_0}\right)\cos\left(\delta_1 + \frac{\pi\Phi}{\Phi_0}\right)}$$

From this, we can see that the inductance of a SQUID can be tuned by applying a magnetic flux. This can be very handy when used in a resonator, because it means that we can tune the resonance frequency.

Note that this inductance is not a magnetic inductance, but rather a kinetic inductance due to the properties of the Josephson junctions. The loop would also have a geometric inductance, which is not considered here as it is not interesting for tuning the SQUID.