Solid state systems for quantum information, Correction 2

Assistants : franco.depalma@epfl.ch, filippo.ferrari@epfl.ch

1 Exercises

Exercise 1 : Coupled LC resonators

Consider two LC resonators, see Fig. 1, with respective inductance, capacitance values L_1, C_1 and L_2, C_2. These resonators are capacitively coupled through a capacitance C_0. The flux variable at the i-th independent node corresponds to ϕ_i.

1. Write down the Lagrangian $\mathcal{L}(\phi, \dot{\phi})$ of the system as a quadratic function of the node flux variables ϕ_i and their derivatives $\dot{\phi}_i$. Introduce the capacitance matrix $\mathbb{C}$ and the inverse of the inductance matrix $\mathbb{L}^{-1}$ and use the flux variables and their derivatives in the vector representation,

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \text{and} \qquad \dot{\vec{\phi}} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

2. Perform the Legendre transformation analytically and extract the Hamiltonian H, as a function of the charge $Q_i = \partial\mathcal{L}(\phi, \dot{\phi})/\partial\dot{\phi}_i$ and the flux variables ϕ_i. Rewrite this Hamiltonian as a quadratic form, using $\mathbb{C}^{-1}$ and $\mathbb{L}^{-1}$.

   Hint: A square 2×2 matrix is inverted by

$$\mathbb{A}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{\det(\mathbb{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

3. To perform quantization, first rewrite the Hamiltonian $\mathcal{H}$ in terms of both inductances L_1, L_2 and the bare (uncoupled) angular frequencies ω_i with $\omega_i^2 = \mathbb{L}_{ii}^{-1}(\mathbb{C}^{-1})_{ii}$. Summarize the capacitive coupling in a single constant $\beta = C_0/\sqrt{(C_1 + C_0)(C_2 + C_0)}$. Perform the quantization by introducing the corresponding quantum operators $\hat{Q}_i$ and $\hat{\phi}_i$ which satisfy the canonical commutation relation $[\hat{Q}_i, \hat{\phi}_j] = -i\hbar\delta_{ij}$. Subsequently, use the following definition of the charge and flux operator to write the Hamiltonian H in terms of annihilation and creation operators, $\hat{a}_i$ and $\hat{a}_i^\dagger$,

$$\hat{Q}_i = -i\sqrt{\frac{\hbar}{2L_i\omega_i}}(\hat{a}_i - \hat{a}_i^\dagger), \qquad \text{and} \qquad \hat{\phi}_i = \sqrt{\frac{\hbar L_i \omega_i}{2}}(\hat{a}_i + \hat{a}_i^\dagger)$$

4. Apply the rotating wave approximation (RWA) on the coupling term and diagonalize the resulting quadratic Hamiltonian with a Bogoliubov transformation. Discuss the physical interpretation of the obtained eigenenergies and eigenmodes.

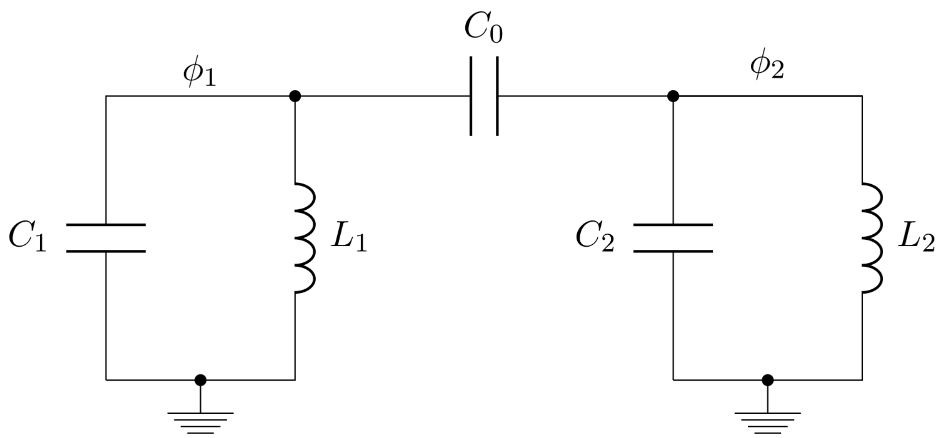

Figure 1: Circuit diagram of two capacitively coupled LC resonators.

Hint: To apply the rotating wave approximation, *suppose* that the Hamiltonian is driven at a frequency ω_d. Perform a unitary transformation in the frame rotating at the pump frequency and understand which terms are fastly oscillating in time. To diagonalize the RWA Hamiltonian, consider a Bogoliubov transformation, which consists in the following ansatz

$$\hat{\alpha} = u\hat{a}_1 + v\hat{a}_2, \tag{1}$$

where u and v are generally complex amplitudes. To find u and v impose $[\hat{H}, \hat{\alpha}] = -E\hat{\alpha}$ and solve the corresponding 2×2 system, and remember that $[\hat{\alpha}, \hat{\alpha}^\dagger] = \mathbb{1}$.

5. Consider now the following values for the capacitances of the two LC oscillators: $C_1 = C_2 = 70\,\mathrm{fF}$, and the following value for the inductance $L_1 = 10\,\mathrm{nH}$. Compute
   - The bare mode frequencies ω_1 and ω_2 as a function of L_2.
   - The coupled mode frequencies of the RWA Hamiltonian according to the formula you have obtained in the previous point, supposing $C_0 = 10\,\mathrm{fF}$.

   Discuss your findings from a physical point of view.

Solution 1 :

1. The Lagrangian is of the form

$$\begin{aligned}
\mathcal{L}(\phi, \dot{\phi}) &= E_{\text{kin}} - E_{\text{pot}} \\
&= \frac{1}{2}\left(C_0(\dot{\phi}_1 - \dot{\phi}_2)^2 + C_1\dot{\phi}_1^2 + C_2\dot{\phi}_2 - \frac{\phi_1^2}{L_1} - \frac{\phi_1^2}{L_2}\right) \\
&= \frac{1}{2}\begin{pmatrix}\dot{\phi}_1 & \dot{\phi}_2\end{pmatrix}\mathbb{C}\begin{pmatrix}\dot{\phi}_1 \\ \dot{\phi}_2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\mathbb{L}^{-\mathbb{1}}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \\
&= \frac{1}{2}\dot{\vec{\phi}}^{\mathrm{T}}\mathbb{C}\dot{\vec{\phi}} - \frac{1}{2}\vec{\phi}^{\mathrm{T}}\mathbb{L}^{-1}\vec{\phi},
\end{aligned}$$

with matrices

$$\mathbb{C} = \begin{pmatrix} C_0 + C_1 & -C_0 \\ -C_0 & C_0 + C_2 \end{pmatrix} \qquad \text{and} \qquad \mathbb{L}^{-1} = \begin{pmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \end{pmatrix}.$$

2. The Legendre transformation is performed by first solving for the charge Q_i at each node,

$$Q_i = \frac{\partial \mathcal{L}(\phi, \dot{\phi})}{\partial \dot{\phi}_i} = \sum_{j=1}^{2} \mathbb{C}_{ij} \dot{\phi}_j = \mathbb{C}_{ij} \dot{\phi}_j,$$

where the expression after the last equal sign using Einstein's summation convention is a shorthand notation for the explicit summation before the last equal sign. In terms of the inverse capacitance matrix and the vector of charges $\vec{Q} = (Q_1, Q_2)^T$, we obtain

$$\dot{\vec{\phi}} = \mathbb{C}^{-1} \vec{Q}.$$

The Hamiltonian $\mathcal{H}$ is then obtained by using the Legendre transformation,

$$\mathcal{H} = \sum_{i=1}^{2} Q_i \dot{\phi}_i - \mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2} \vec{Q}^{\mathrm{T}} \mathbb{C}^{-1} \vec{Q} + \frac{1}{2} \vec{\phi}^{\mathrm{T}} \mathbb{L}^{-1} \vec{\phi}$$

with the inverse of both capacitance and inductance matrices,

$$\mathbb{C}^{-1} = \frac{1}{C_1 C_2 + C_0 (C_1 + C_2)} \begin{pmatrix} C_0 + C_2 & C_0 \\ C_0 & C_0 + C_1 \end{pmatrix} \quad \text{and} \quad \mathbb{L}^{-1} = \begin{pmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \end{pmatrix}.$$

3. We write out the bare angular mode frequencies ω_i,

$$\omega_1 = \sqrt{\frac{1}{L_1} \frac{C_2 + C_0}{C_1 C_2 + C_0 C_1 + C_0 C_2}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{1}{L_2} \frac{C_1 + C_0}{C_1 C_2 + C_0 C_1 + C_0 C_2}},$$

and rewrite the inverse capacitance matrix in terms of inductances L_i, bare angular frequencies ω_i and the coupling constant β,

$$\mathbb{C}^{-1} = \begin{pmatrix} L_1 \omega_1^2 & \beta \sqrt{L_1 L_2} \omega_1 \omega_2 \\ \beta \sqrt{L_1 L_2} \omega_1 \omega_2 & L_2 \omega_2^2 \end{pmatrix}.$$

Now we are able to formulate the Hamiltonian $\mathcal{H}$ as follows,

$$\mathcal{H} = \frac{1}{2} L_1 \omega_1^2 Q_1^2 + \frac{1}{2L_1} \phi_1^2 + \frac{1}{2} L_2 \omega_2^2 Q_2^2 + \frac{1}{2L_2} \phi_2^2 + \beta \sqrt{L_1 L_2} \omega_1 \omega_2 Q_1 Q_2$$

We perform quantization by replacing the flux and charge variables with their corresponding quantum operators $\hat{\phi}$ and $\hat{Q}$. Subsequently, we replace those operators with their definition in terms of annihilation and creation operators as seen in the class. After simplifying the resulting expression and obeying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we obtain the Hamiltonian in second quantization ($\hbar = 1$)

$$\hat{H} = \hbar \omega_1 \left(\hat{a}_1^\dagger \hat{a}_1 + 1/2 \right) + \hbar \omega_2 \left(\hat{a}_1^\dagger \hat{a}_1 + 1/2 \right) - \hbar \frac{\beta}{2} \sqrt{\omega_1 \omega_2} \left(\hat{a}_1 - \hat{a}_1^\dagger \right) \left(\hat{a}_2 - \hat{a}_2^\dagger \right)$$

we note that in the limit of small coupling capacitance C_0

$$\omega_i \simeq \frac{1}{\sqrt{L_i\left(C_i+C_0\right)}} \qquad \text{for} \qquad C_0 \ll C_1, C_2$$

and the coupling strength approximates to

$$g = -\frac{\beta}{2}\sqrt{\omega_1\omega_2} \simeq -\frac{1}{2}\frac{C_0}{\sqrt{C_1 C2}}\sqrt{\omega_1\omega_2} \qquad \text{for} \qquad C_0 \ll C_1, C_2. \tag{2}$$

4. The quadratic bosonic Hamiltonian reads (we take $\hbar = 1$ and we neglect the constant terms in the Hamiltonian, which do not have any influence on the dynamics)

$$\hat{H} = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 - g\left(\hat{a}_1 - \hat{a}_1^\dagger\right)\left(\hat{a}_2 - \hat{a}_2^\dagger\right). \tag{3}$$

The rotating wave approximation can be carried out by neglecting the counter-rotating terms which oscillate faster than the system's frequency. If we imagine to drive the system at a certain frequency ω_d, we can then suppose the following unitary

$$\hat{U} = e^{i\omega_d t\left(a_1^\dagger \hat{a}_1 + a_2^\dagger \hat{a}_2\right)}. \tag{4}$$

To find the rotated Hamiltonian it is sufficient to rotate single bosonic operators. We consider for example $\hat{a}_1$. By applying the Baker-Campbell-Haussdorff formula we obtain

$$\begin{aligned} \hat{U}\hat{a}_1\hat{U}^\dagger = e^{i\omega_d t \hat{a}_1^\dagger \hat{a}_1}\hat{a}_1 e^{-i\omega_d t \hat{a}_1^\dagger \hat{a}_1} &= \hat{a}_1 + i\omega_d t[\hat{a}_1^\dagger \hat{a}_1, \hat{a}_1] + \frac{1}{2}(i\omega_d t)^2[\hat{a}_1^\dagger \hat{a}_1, [\hat{a}_1^\dagger \hat{a}_1, \hat{a}_1]] + ... \\ &= \hat{a}_1 - (i\omega_d t)\hat{a}_1 + \frac{1}{2}(i\omega_d t)^2 \hat{a}_1 + ... = \sum_{k=0}^{+\infty}\frac{(-1)^n}{n!}(i\omega_d t)^n \hat{a}_1 = \hat{a}_1 e^{-i\omega_d t}. \end{aligned} \tag{5}$$

Similarly, we find $\hat{U}\hat{a}_2\hat{U}^\dagger = \hat{a}_2 e^{-i\omega_d t}$. We then notice that terms containing an equal number of creation and annihilation operators do not oscillate in the frame rotating at the pump frequency, while terms like $\hat{a}_1\hat{a}_2$ and $\hat{a}_1^\dagger\hat{a}_2^\dagger$ oscillates at $2\omega_d$. Those are the fastly oscillating terms that can be neglected within the RWA and we arrive at the RWA Hamiltonian

$$\hat{H} = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + g\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right). \tag{6}$$

The above Hamiltonian is again quadratic, but can be diagonalized in a much simpler way with respect to the previous Hamiltonian containing counter-rotating terms. The method we adopt to diagonalize the Hamiltonian is a Bogoliubov transformation, which consists in writing a linear combination of the two bosonic operators

$$\hat{\alpha} = u\hat{a}_1 + v\hat{a}_2. \tag{7}$$

If the Hamiltonian were diagonal in the $\hat{\alpha}$ operators, then we would have that $[\hat{H}, \hat{\alpha}] = -E\hat{\alpha}$ with E the eigenenergies of the system. If we compute the commutator explicitly we get

$$[\hat{H}, \hat{\alpha}] = -u\omega_1\hat{a}_1 - v\omega_2\hat{a}_2 - ug\hat{a}_2 - vg\hat{a}_1 = -Eu\hat{a}_1 - Ev\hat{a}_2 = -E\hat{\alpha}. \tag{8}$$

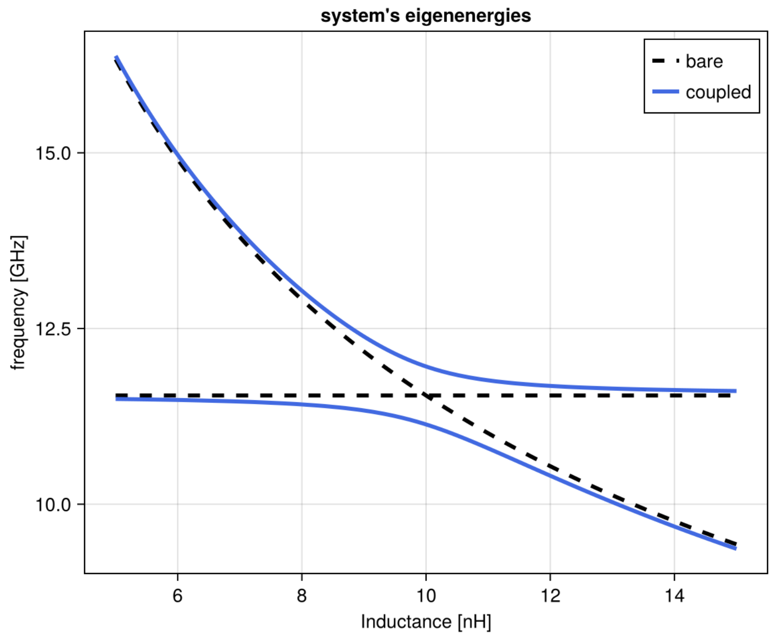

Figure 2: Bare (black-dashed line) and dressed (purple continuous line) eigenfrequencies of the coupled LC circuit system as a function of the inductance of the second LC resonator.

The above equation can be recast in the following 2×2 system

$$\begin{cases} \omega_1 u + g v = E u \\ g u + \omega_2 v = E v \end{cases} \qquad \Longrightarrow \qquad \begin{pmatrix} \omega_1 & g \\ g & \omega_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} v \\ v \end{pmatrix}. \tag{9}$$

The eigenvalues can be obtained from the characteristic equation

$$\begin{pmatrix} \omega_1 - E & g \\ g & \omega_2 - E \end{pmatrix} = (\omega_1 - E)(\omega_2 - E) - g^2 = 0, \tag{10}$$

whose solutions reads

$$E_\pm = \frac{\omega_1 + \omega_2}{2} \pm \sqrt{\left(\frac{\omega_1 - \omega_2}{2}\right)^2 + g^2}. \tag{11}$$

Notice that if $g = 0$ the two solutions are simply $E_\pm = \omega_{1,2}$, since the Hamiltonian is already diagonal. To find the eigenvectors we use the fact that $\hat{\alpha}$ is a bosonic operator and

$$[\hat{\alpha}, \hat{\alpha}^\dagger] = [u\hat{a}_1 + v\hat{a}_2, u\hat{a}_1^\dagger + v\hat{a}_2^\dagger] = u^2 + v^2 = 1. \tag{12}$$

From the 2×2 system for the Bogoliubov amplitudes we get

$$v = \frac{E - \omega_1}{g} u, \tag{13}$$

and from the condition $u^2 + v^2 = 1$ we finally obtain

$$u = \frac{\pm g}{\sqrt{g^2 + (E_\pm - \omega_1)^2}}, \qquad v = \frac{E_\pm - \omega_1}{\sqrt{g^2 + (E_\pm - \omega_1)^2}}. \tag{14}$$

The Hamiltonian can finally be written in the diagonal form

$$\hat{H} = E_+ \hat{\alpha}_+^\dagger \hat{\alpha}_+ + E_- \hat{\alpha}_-^\dagger \hat{\alpha}_-, \tag{15}$$

where $E_\pm$ are the Bogoliubov quasi-energies obtained above. From a physical point of view the diagonal Hamiltonian in terms of modes which are linear combinations of $\hat{a}_1$ and $\hat{a}_2$ indicates that the coupling hybridizes the modes. Notably, the Bogoliubov quasi-energies $E_\pm$ exhibit now an avoided level crossing (or level repulsion), a typical signature of many-body interactions. We study in detail the avoided level crossing in the next point.

5. We consider the values for $C_{0,1,2}$ and $L_{1,2}$ given in the text. Since $C_0 \ll C_{1,2}$ we fix the bare frequencies to $\omega_j \simeq 1/\sqrt{L_j(C_0 + C_j)}$ and the coupling to $g \simeq -(\sqrt{\omega_1\omega_2}/2) \times C_0/\sqrt{C_1 C_2}$. We plot the results in Fig. 2. The bare frequencies cross at a certain value of L_2, while the dressed frequencies exhibit an avoided level crossing, as predicted by the Bogoliubov diagonalization.

Exercise 2 : Circuit Quantization

1. The goal of this exercise is to find the Lagrangian and quantized Hamiltonian of a LC resonator capacitively coupled to a time variable voltage source (see Fig. 3a).

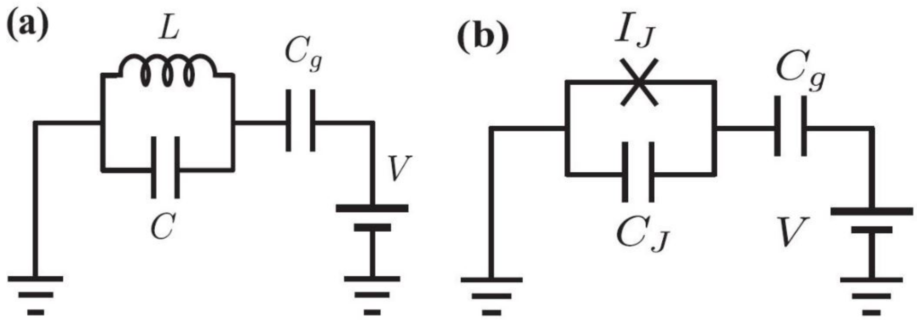

Figure 3: a) Equivalent circuit for an LC resonator consisting on an inductor in parallel with a capacitor, subject to an external potential V(t). b) Equivalent circuit for a non-linear inductor (a Josephson junction) in parallel with a capacitor, subject to an external potential V(t).

   (a) First, decompose the circuit to identify the branch and node fluxes. Find the relation between them. You should end up with only one flux variable.

   (b) Find the Lagrangian of the system.

   (c) With the Legendre transformation, find the associated quantized Hamiltonian.

2. The circuit in Fig. 3b models a Cooper Pair Box or equivalently, as you will see in future sessions, a transmon qubit. It mirrors the LC resonator of the previous point, but now the inductor has been replaced by a Josephson junction, effectively behaving as a non-linear inductor.

   (a) Find the Lagrangian of the CPB.

   (b) Find the associated quantized Hamiltonian.

Solution 2 :

1. (a) We see that the branch flux associated with the inductor ϕ_a is the same than the one associated with ϕ_d as the two components are in parallel. The flux ϕ_0 is associated to the ground plane so that we can take $\phi_0 = 0$ as our reference. In the end:

$$\phi_a = \phi_d = \phi_1 - \phi_0 = \phi_1$$
$$\phi_b = \phi_2 - \phi_1$$
$$\phi_c = \phi_2 - \phi_0 = -\phi_2$$

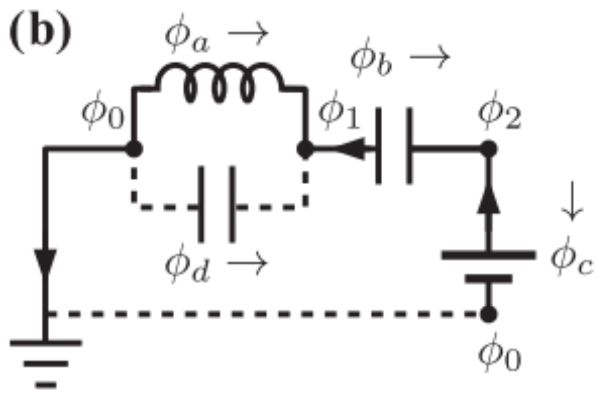

Figure 4: Nodes, node fluxes, and branch fluxes for the circuit description and quantization.

(b)

$$\mathcal{L} = \frac{1}{2}C\dot{\phi_a}^2 + \frac{1}{2}C_g\dot{\phi_b}^2 - \frac{1}{2L}\phi_a^2$$
$$\mathcal{L} = \frac{1}{2}C\dot{\phi_1}^2 + \frac{1}{2}C_g(\dot{\phi_2} - \dot{\phi_1})^2 - \frac{1}{2L}\phi_1^2$$

We can use Kirchhoff's law for the voltages to write:

$$\dot{\phi}_a + \dot{\phi}_b + \dot{\phi}_c = \dot{\phi}_1 + (\dot{\phi}_2 - \dot{\phi}_1) - V = 0$$

and to deduce that:

$$\dot{\phi}_2 = V$$

Which allows us to finally write the Lagrangian of the system in only one variable:

$$\mathcal{L} = \frac{1}{2}C\dot{\phi_1}^2 + \frac{1}{2}C_g(V - \dot{\phi}_1)^2 - \frac{1}{2L}\phi_1^2$$

The $\frac{1}{2}C_gV^2$ term is a constant term that does not contribute in the dynamics of the system so it can be neglected. Finally:

$$\mathcal{L} = \frac{1}{2}C_\Sigma\dot{\phi_1}^2 - C_gV\dot{\phi}_1 - \frac{1}{2L}\phi_1^2. \tag{16}$$

where $C_\Sigma = C + C_g$

(c) By finding the generalised charge $q_1 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = C_\Sigma\dot{\phi}_1 - C_gV$ and writing the Legendre transformation:

$$H = q_1\dot{\phi}_1 - \mathcal{L}. \tag{17}$$

We find the quantized (we already know that charge and flux are conjugate variables) Hamiltonian:

$$\hat{H} = \frac{1}{2C_\Sigma}(\hat{q_1} - q_g)^2 + \frac{1}{2L}\hat{\phi_1}^2. \tag{18}$$

where $q_g = -C_g V$ is the charge offset due to the generator.

2. (a) Following what was done in the previous exercise, the action of the capacitors does not change when compared to the LC resonator. The inductance contribution becomes non linear, dictated by the phase change across the junction:

$$\mathcal{L} = \frac{1}{2}C_J\dot{\phi}_1^2 + \frac{1}{2}C_g(V - \dot{\phi}_1)^2 + E_J\cos(\phi_1/\Phi_0)$$
$$\mathcal{L} = \frac{1}{2}C_\Sigma\dot{\phi}_1^2 - C_g\dot{\phi}_1 V + E_J\cos(\phi_1/\Phi_0).$$

(b) Once again, we compute the charge of the system $q = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = C_\Sigma\dot{\phi}_1 - C_g V$ to finally find the Hamiltonian of the CPB in quantized form using Legendre transformation $H = q\dot{\phi}_1 - \mathcal{L}$:

$$\hat{H} = \frac{1}{2C_\Sigma}(\hat{q} - q_g)^2 - E_J\cos\left(\hat{\phi}_1/\Phi_0\right)$$
$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J\cos\left(\hat{\delta}\right).$$

where $E_C = \frac{e^2}{2C_\Sigma}$ is the charging energy, e the electron charge, E_J the Josephson energy, $\hat{\delta}$ the phase operator, $\hat{n} = \frac{\hat{q}}{2e}$ the (normalized) charge operator and $n_g = \frac{\hat{q_g}}{2e}$ the (normalized) charge offset.

Exercise 3 : Lossless transmission line

We now look into a very interesting example: a lossless transmission line (TL). A TL allows RF signals to be transmitted without significant amount of losses thanks to its property of confining electromagnetic fields between a central conductor and grounded outer shell (this is the case of the well known coaxial cable, for example). A TL can be modeled with an infinite series of fundamental cells constituted by an inductor within the inner conductor and a capacitor from the inner conductor to ground (see Fig. 5).

The inductors model the inertia against changes in the electric current while the capacitors account for the electrostatic energy stored in the waveguide. The circuit is a discretized version of the guide where each capacitor and inductor accounts for a small segment Δx that is much smaller than the guided wavelengths. The properties of these elements depend on the capacitance and inductance per unit length:

$$C_i = c_i \Delta x, L_i = l_i \Delta x. \tag{19}$$

The goal is again to find the Lagrangian and quantized Hamiltonian for such an equivalent circuit describing a TL.

1. Find the Lagrangian of the system. Consider the branch fluxes along the inductors leftward oriented, meaning $\phi_{i+1\to i} = \phi_{i+1} - \phi_i$.
2. With the Legendre transformation, find the associated quantized Hamiltonian. Now, express it also in terms of c_i and l_i.

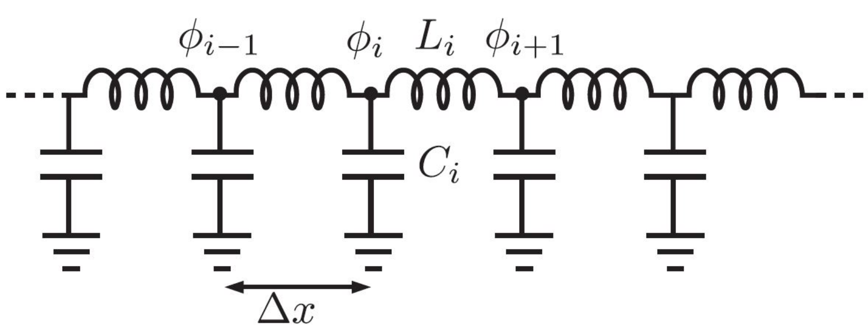

Figure 5: Circuit for a lossless transmission line.

Solution 3 :

1. We have seen from the text that the TL is constituted by identical unitary cells (an inductor L_i within the inner conductor and a capacitor C_i to ground), meaning that to solve the problem we can focus on a single cell only. If we consider the $i-$th cell, the kinetic term of the Lagrangian comes from the capacitor, where the time derivative of the flux to be considered is the $i-$th one $\dot{\phi}_i$, since C_i lies between the $i-$th node and the ground, whereas

the potential one is accounted by the inductor L_i with its corresponding branch flux ϕ_{i+1} - ϕ_i:

$$\mathcal{L}_i = \frac{1}{2}C_i\dot{\phi_i}^2 - \frac{1}{2L_i}(\phi_{i+1} - \phi_i)^2.$$

To get the full Lagrangian of the system, we just need to sum over the total number of cells N:

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2}C_i\dot{\phi_i}^2 - \sum_{i=1}^{N-1} \frac{1}{2L_i}(\phi_{i+1} - \phi_i)^2.$$

2. The generalized charges follow $q_i = \partial\mathcal{L}_i/\partial\dot{\phi}_i$, allowing us to write the full Hamiltonian using the usual Legendre transformation:

$$H = \sum_{i=1}^{N} q_i\dot{\phi}_i - \mathcal{L} = \sum_{i=1}^{N} \frac{1}{2C_i}q_i^2 + \sum_{i=1}^{N-1} \frac{1}{2L_i}(\phi_{i+1} - \phi_i)^2. \tag{20}$$

We can finally quantize H since, as we have already seen many times, charge and flux are conjugate variables, i.e. $[\hat{\phi}_i, \hat{q}_i] = i\hbar\delta_{ij}$. We can express $\hat{H}$ as a function of c_i and l_i:

$$\hat{H} = \sum_{i=1}^{N} \frac{\Delta x}{2c_i(x_i)} \left(\frac{\hat{q}_i}{\Delta x}\right)^2 + \sum_{i=1}^{N-1} \frac{\Delta x}{2l_i(x_i)} \left(\frac{\hat{\phi}_{i+1} - \hat{\phi}_i}{\Delta x}\right)^2. \tag{21}$$

In conclusion, we can introduce the charge density operator $\hat{\rho}(x_i) = \hat{q}_i/\Delta x$ and replace sums with integral:

$$\hat{H} = \int dx \left(\frac{1}{2c(x)}\hat{\rho}(x)^2 + \frac{1}{2l(x)}[\partial_x\hat{\phi}(x)]^2\right). \tag{22}$$