Solid state systems for quantum information, Correction 1

Assistants : franco.depalma@epfl.ch, filippo.ferrari@epfl.ch

1 Exercises

Exercise 1 : Pauli algebra and the Bloch sphere representation

In this exercise we deal with the fundamental properties of quantum mechanical two-level systems. We study the algebra obeyed by two-level system operators (the Pauli algebra) and we understand how to represent two-level systems on the Bloch sphere, considering both pure and mixed states.

A single qubit state can be represented by its Bloch vector $\boldsymbol{r} = (\langle\hat{\sigma}_x\rangle, \langle\hat{\sigma}_y\rangle, \langle\hat{\sigma}_z\rangle)^T$, which is defined on three-dimensional unit sphere (i.e. $|\boldsymbol{r}| \leq 1$). Where $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the Pauli operators defined as:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Show that these matrices are Hermitian and unitary.
2. Show that the Pauli matrices satisfy the following commutation relation: $[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$, and find similar expressions for $[\hat{\sigma}_y, \hat{\sigma}_z]$ and $[\hat{\sigma}_z, \hat{\sigma}_x]$.
3. Show that an arbitrary pure state, parametrized as $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin\theta/2 e^{i\phi}|1\rangle$ is represented by a Bloch vector of unit length. Graphically visualize the Bloch vector on the Bloch sphere for $\theta = \pi/4$ and $\phi = \pi/2$.
4. Calculate the length of the Bloch vector for a mixed state with density matrix $\hat{\rho} = (1 - \varepsilon)|\psi\rangle\langle\psi| + \frac{\varepsilon}{2}\mathbb{1}$.

Solution 1 :

1. A matrix, A, is *Hermitian* if $A = A^\dagger$ where the dagger symbol, †, is the transpose conjugate. It is *unitary* if $AA^\dagger = \mathbb{1}$. We detail the calculation only for $\hat{\sigma}_y$. To verify that $\hat{\sigma}_y$ is Hermitian it is sufficient to compute

$$\hat{\sigma}_y^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^\star = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{1}$$

To verify that $\hat{\sigma}_y$ is unitary it is sufficient to compute

$$\hat{\sigma}_y\hat{\sigma}_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hat{\sigma}_y^2 = \mathbb{1}. \tag{2}$$

In the same way we can verify Hermiticity and unitarity of $\hat{\sigma}_x$ and $\hat{\sigma}_z$.

2. To show that $[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$ we can simply bruteforce calculate it:

$$[\hat{\sigma}_x, \hat{\sigma}_y] = \hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\hat{\sigma}_z \tag{3}$$

We proceed with the same calculations for the two others commutation relations and we find $[\hat{\sigma}_y, \hat{\sigma}_z] = 2i\hat{\sigma}_x$ and $[\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$.

3. Given a pure state, the expectation value of an operator $\hat{O}$ is $\langle\hat{O}\rangle = \langle\psi|\,\hat{O}\,|\psi\rangle$. Here $|\psi\rangle = (\cos(\theta/2),\ \sin(\theta/2)e^{i\phi})^{\mathrm{T}}$. We have

$$\langle\hat{\sigma}_x\rangle = \langle\psi|\,\hat{\sigma}_x\,|\psi\rangle = \begin{pmatrix}\cos(\theta/2) & \sin(\theta/2)e^{-i\phi}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix}\cos(\theta/2) \\ \sin(\theta/2)e^{i\phi}\end{pmatrix} \tag{4}$$

$$= \sin(\theta/2)\cos(\theta/2)\left(e^{-i\phi} + e^{i\phi}\right) = \sin(\theta)\cos(\phi), \tag{5}$$

where we have used $e^{-i\phi} + e^{i\phi} = 2\mathrm{Re}\left(e^{i\phi}\right) = 2\cos(\phi)$ and $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$.
Similarly, $\langle\hat{\sigma}_y\rangle = \sin(\theta)\sin(\phi)$, and $\langle\hat{\sigma}_z\rangle = \cos(\theta)$. The magnitude of the vector is given by $|\boldsymbol{r}| = \sqrt{\langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2} = 1$.

The state with $\theta = \pi/4$ and $\phi = \pi/2$, lies in the yz-plane, 45 degrees towards the "south" pole of the Bloch sphere. This can also be seen by calculating $\langle\hat{\sigma}_x\rangle = 0$, $\langle\hat{\sigma}_y\rangle = 1/\sqrt{2}$ and $\langle\hat{\sigma}_z\rangle = 1/\sqrt{2}$.

4. Given a mixed state, the expectation value of an operator $\hat{O}$ on a density matrix $\hat{\rho}$ is given by $\langle\hat{O}\rangle = \mathrm{Tr}(\hat{\rho}\hat{O})$. The we consider is

$$\hat{\rho} = (1-\varepsilon)\,|\psi\rangle\langle\psi| + \frac{\varepsilon}{2}\mathbb{1} \tag{6}$$

which is a mixture of the pure state $|\psi\rangle$ of the previous point, for which we have

$$|\boldsymbol{r}_\psi|^2 = 1 = \mathrm{Tr}(\hat{\sigma}_x\,|\psi\rangle\langle\psi|)^2 + \mathrm{Tr}(\hat{\sigma}_y\,|\psi\rangle\langle\psi|)^2 + \mathrm{Tr}(\hat{\sigma}_z\,|\psi\rangle\langle\psi|)^2, \tag{7}$$

and of the identity matrix. For the latter, it is useful to note that the Pauli matrices have all a null trace $\mathrm{Tr}(\hat{\sigma}_x\mathbb{1}) = \mathrm{Tr}(\hat{\sigma}_y\mathbb{1}) = \mathrm{Tr}(\hat{\sigma}_z\mathbb{1}) = 0$. From the linearity of the trace, we conclude that $\mathrm{Tr}(\hat{\sigma}_j\rho) = (1-\varepsilon)\,\mathrm{Tr}(\hat{\sigma}_j\,|\psi\rangle\langle\psi|) + \frac{\varepsilon}{2}\,\mathrm{Tr}(\hat{\sigma}_j\mathbb{1}) = (1-\varepsilon)\,\mathrm{Tr}(\hat{\sigma}_j\,|\psi\rangle\langle\psi|)$ for any Pauli operator $\hat{\sigma}_j$ $(j = x, y, z)$. It follows that the length of the Bloch vector of that state is given by

$$|\boldsymbol{r}_\rho| = \sqrt{(1-\varepsilon)^2\langle\psi|\,\hat{\sigma}_x\,|\psi\rangle^2 + (1-\varepsilon)^2\langle\psi|\,\hat{\sigma}_y\,|\psi\rangle^2 + (1-\varepsilon)^2\langle\psi|\,\hat{\sigma}_z\,|\psi\rangle^2} \tag{8}$$

$$= (1-\varepsilon)\sqrt{\langle\psi|\,\hat{\sigma}_x\,|\psi\rangle^2 + \langle\psi|\,\hat{\sigma}_y\,|\psi\rangle^2 + \langle\psi|\,\hat{\sigma}_z\,|\psi\rangle^2} = (1-\varepsilon)\,|\boldsymbol{r}| = (1-\varepsilon) \in [0,1].$$

Exercise 2 : Dynamics of a two-level system in a magnetic field

In this exercise we consider the dynamics of a two-level system in the presence of an external magnetic field. The Hamiltonian reads

$$\hat{H} = -\frac{\hbar\gamma}{2}\boldsymbol{B}\cdot\hat{\boldsymbol{\sigma}} = -\frac{\hbar\gamma}{2}(B_x\hat{\sigma}_x + B_y\hat{\sigma}_y + B_z\hat{\sigma}_z), \tag{9}$$

where γ is the gyromagnetic ratio. The system's dynamics is governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle, \tag{10}$$

whose general solution is given by

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle. \tag{11}$$

1. Consider now the case where we have a constant magnetic field in the z-direction:

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix} \tag{12}$$

   (a) If at time $t = 0$ the state is given by $|\psi(0)\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$, calculate the state $|\psi(t)\rangle$ after a given time $t = t_1$.
   Hint: Use the following identity $e^{i\frac{\theta}{2}(n_x\hat{\sigma}_x+n_y\hat{\sigma}_y+n_z\hat{\sigma}_z)} = \cos(\theta/2)\mathbb{1} + i\sin(\theta/2)(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z)$.

2. Let's assume we have a constant magnetic field in the x-direction:

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ 0 \\ 0 \end{pmatrix}$$

   (a) Calculate the state after a time t assuming that at $t = 0$ the state was in $|\psi(0)\rangle = |0\rangle$. What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$ (where $\omega = \gamma B_x$) ?
   (b) Calculate the state after a time t assuming that at $t = 0$ the state was in $|\psi(0)\rangle = |1\rangle$.What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$?

Solution 2 :

1. In order to find $|\psi(t)\rangle$ at a time t_1, we apply the identity given in the hint.

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{i\frac{\omega t}{2}\hat{\sigma}_z} = \cos\left(\frac{\omega t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega t}{2}\right)\hat{\sigma}_z = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) + i\sin\left(\frac{\omega t}{2}\right) & 0 \\ 0 & \cos\left(\frac{\omega t}{2}\right) - i\sin\left(\frac{\omega t}{2}\right) \end{pmatrix}$$
$$= \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}} \end{pmatrix} = e^{i\frac{\omega t}{2}}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\omega t} \end{pmatrix}, \tag{13}$$

and thus

$$|\psi(t)\rangle = e^{\frac{i\omega t}{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} |\psi(0)\rangle \,. \tag{14}$$

where $\omega = B_z\gamma$. Using the state at time $t = 0$, $|\psi(0)\rangle = \cos\theta/2\,|0\rangle + \sin\theta/2 e^{i\phi}\,|1\rangle$, we have

$$\begin{aligned} |\psi(t)\rangle &= e^{\frac{i\omega t}{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \left[\cos{(\theta/2)}\,|0\rangle + \sin{(\theta/2)} e^{i\phi}\,|1\rangle\right] \\ &= e^{\frac{i\omega t}{2}} \left[\cos{(\theta/2)}\,|0\rangle + \sin{(\theta/2)} e^{i\phi - i\omega t}\,|1\rangle\right]. \end{aligned}$$

$e^{\frac{i\omega t}{2}}$ can be neglected as it is a global phase factor, having no influence on expectation values. Hence we find

$$|\psi(t)\rangle = \cos{(\theta/2)}\,|0\rangle + \sin{(\theta/2)} e^{i\phi - i\omega t}\,|1\rangle \,.$$

Applying a $\boldsymbol{B}$-field along the z-axis, makes the spin precess along the z axis at the angular frequency ω.

2. For the case with $\boldsymbol{B}$ along the x direction we can apply the same process.

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{i\frac{\omega t}{2}\hat{\sigma}_x} = \cos\left(\frac{\omega t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega t}{2}\right)\hat{\sigma}_x = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix}, \tag{15}$$

and thus

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} |\psi(0)\rangle \,. \tag{16}$$

Let's now derive $|\psi(t)\rangle$ for the two cases.

(a) $|\psi(0)\rangle = |0\rangle$; the time-evolved state reads

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} |0\rangle = \cos\left(\frac{\omega t}{2}\right)|0\rangle + i\sin\left(\frac{\omega t}{2}\right)|1\rangle \,,$$

For $t = \pi/\omega$ we have

$$|\psi(t = \pi/\omega)\rangle\rangle = i\,|1\rangle \,,$$

so the time evolution leads to a population transfer from $|0\rangle$ to $|1\rangle$. For $t = \pi/2\omega$ we have

$$|\psi(t = \pi/2\omega)\rangle\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right),$$

so the time evolution brings $|0\rangle$ to a quantum superposition of states.

(b) $|\psi(0)\rangle = |1\rangle$; the time-evolved state reads

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} |1\rangle = i\sin\left(\frac{\omega t}{2}\right)|0\rangle + \cos\left(\frac{\omega t}{2}\right)|1\rangle \,.$$

For $t = \pi/\omega$ we have

$$|\psi(t = \pi/\omega)\rangle\rangle = i\,|0\rangle\,,$$

so the time evolution leads to a population transfer from $|1\rangle$ to $|0\rangle$. For $t = \pi/2\omega$ we have

$$|\psi(t = \pi/2\omega)\rangle\rangle = \frac{1}{\sqrt{2}}\,(i\,|0\rangle + |1\rangle)\,,$$

so the time evolution brings $|1\rangle$ to a quantum superposition of states.

Exercise 3 : Driven two-level system

In this exercise we study the dynamics of a driven two-level system. The time-dependent Hamiltonian describing a single two-level system subject to a continuous drive at frequency ω_d is (assuming $\hbar = 1$)

$$\hat{H} = \frac{\omega_0}{2}\hat{\sigma}_z + \Omega\hat{\sigma}_x \cos(\omega_d t). \tag{17}$$

1. Write the Hamiltonian in the frame rotating at the drive frequency ω_d and apply the rotating-wave approximation to eliminate the remaining time-dependence.

2. Diagonalize the time-independent Hamiltonian obtaining the eigenvalues and the eigenvectors. Hint: express the eigenvectors in terms of the rotation angle θ, where

$$\tan\theta = \frac{\Omega}{\Delta}, \qquad \sin\theta = \frac{\Omega}{\sqrt{\Delta^2+\Omega^2}}, \qquad \cos\theta = \frac{\Delta}{\sqrt{\Delta^2+\Omega^2}},$$

   where $\Delta = \omega_0 - \omega_d$ is the qubit-to-pump detuning.

3. Calculate the probability of being in the excited state $|1\rangle$ under the assumption that at time zero $t = 0$ the system is in the state $|\psi(0)\rangle = |0\rangle$

$$P_1(t) = |\langle 1|\psi(t)\rangle|^2 \tag{18}$$

4. Calculate the time average of the probability of being in the excited state

$$\bar{P}_1 = \lim_{t\to\infty}\frac{1}{T}\int_0^T P_1(t)dt \tag{19}$$

5. Plot the Rabi frequency [detuning of the oscillation of $P_1(t)$] as a function of time t.

Solution 3 :

1. We consider the Hamiltonian of the driven two-level system given in the text

$$\hat{H} = \frac{\omega_0}{2}\hat{\sigma}_z + \Omega\hat{\sigma}_x \cos(\omega_d t). \tag{20}$$

   We move to a frame rotating at the drive frequency (ω_d) using the unitary transformation (as explained in class) $\hat{U}(t) = e^{i\omega_d t\frac{\hat{\sigma}_z}{2}}$. The transformed Hamiltonian in the rotating frame is given by the formula

$$\hat{H} \to \hat{U}\hat{H}\hat{U}^\dagger + i\frac{\partial\hat{U}}{\partial t}\hat{U}^\dagger. \tag{21}$$

   We first compute the second term in the above equation, which is

$$i\frac{\partial\hat{U}}{\partial t}\hat{U}^\dagger = -\frac{\omega_d}{2}\hat{\sigma}_z. \tag{22}$$

Now we use the formula from the previous exercise to transform $\hat{\sigma}_x$

$$\hat{U}\hat{\sigma}_x\hat{U}^\dagger = \hat{\sigma}_x \cos(\omega_d t) - \hat{\sigma}_y \sin(\omega_d t). \tag{23}$$

Notice that the same result can be obtained by using the Baker-Campbell-Haussdorff formula. Finally, we notice that $\hat{U}$ commutes with $\hat{\sigma}_z$, and the transformed Hamiltonian reads

$$\hat{H} = \frac{\Delta}{2}\hat{\sigma}_z + \Omega\left[\hat{\sigma}_x \cos(\omega_d t) - \hat{\sigma}_y \sin(\omega_d t)\right]\cos(\omega_d t), \tag{24}$$

where $\Delta = \omega_0 - \omega_d$ is the qubit-to-pump detuning. To simplify the above equation, we can use the following trigonometric identities

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)], \tag{25}$$
$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)],$$

and we finally obtain

$$\hat{H}_R = \frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega}{2}\left[\hat{\sigma}_x(\cos 2\omega_d t + 1) - \hat{\sigma}_y \sin 2\omega_d t\right]. \tag{26}$$

We now apply the rotating wave approximation (RWA). The terms oscillating at $2\omega_d$ are rapidly varying and average to zero over time. Keeping only the time-independent terms, we obtain the effective Hamiltonian:

$$\hat{H} = \frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega}{2}\hat{\sigma}_x. \tag{27}$$

This describes the effective dynamics of the system within the RWA.

2. We first express the RWA Hamiltonian in matrix form as

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{bmatrix}. \tag{28}$$

To find the eigenvalues, it is sufficient to solve the secular equation

$$\det\left(\mathcal{H} - \varepsilon I\right) = \begin{vmatrix} \frac{\Delta}{2} - \varepsilon & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\Delta}{2} - \varepsilon \end{vmatrix} = 0, \tag{29}$$

which yields the equation

$$\varepsilon^2 - \frac{\Delta^2}{4} - \frac{\Omega^2}{4} = 0, \tag{30}$$

whose solutions read

$$\varepsilon_\pm = \pm\frac{\sqrt{\Delta^2 + \Omega^2}}{2}. \tag{31}$$

Now we find the eigenvectors. We detail the procedure for the eigenvector associated to $E_+ = \frac{\sqrt{\Delta^2+\Omega^2}}{2}$. We need to solve the 2×2 system

$$\begin{bmatrix} \frac{\Delta}{2} - \frac{\sqrt{\Delta^2+\Omega^2}}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\Delta}{2} - \frac{\sqrt{\Delta^2+\Omega^2}}{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{32}$$

If we set the first row to zero we can write down the equation

$$\left(\frac{\Delta}{2} - \frac{\sqrt{\Delta^2 + \Omega^2}}{2}\right) a + \frac{\Omega}{2} b = 0, \tag{33}$$

which gives

$$a = \frac{\Omega}{\sqrt{\Delta^2 + \Omega^2} - \Delta} b. \tag{34}$$

This leads to the eigenvector

$$|+\rangle = \begin{pmatrix} \frac{\Omega}{\sqrt{\Delta^2+\Omega^2}-\Delta} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ 1 \end{pmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{35}$$

where we used the relations given in the main text

$$\tan\theta = \frac{\Omega}{\Delta}, \quad \cos\theta = \frac{\Delta}{\sqrt{\Delta^2 + \Omega^2}}, \quad \sin\theta = \frac{\Omega}{\sqrt{\Delta^2 + \Omega^2}}, \tag{36}$$

Similarly, one gets for the second eigenvector

$$|-\rangle = -\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle. \tag{37}$$

3. We first invert the relation between the $\{|+\rangle, |-\rangle\}$ eigenstates and the $\{|1\rangle, |0\rangle\}$ states and we get the following expression for $|0\rangle$:

$$|0\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle - \sin\left(\frac{\theta}{2}\right)|-\rangle. \tag{38}$$

At this point is easy to compute the time evolution generated by $\hat{H}$:

$$|\psi(t)\rangle = e^{-i\hat{H}t}|0\rangle = e^{-i\varepsilon_+ t}\cos\left(\frac{\theta}{2}\right)|+\rangle - e^{i\varepsilon_+ t}\sin\left(\frac{\theta}{2}\right)|-\rangle. \tag{39}$$

The overlap with $|1\rangle$ is then given by

$$\langle 1|\psi(t)\rangle = e^{-i\varepsilon_+ t}\cos\left(\frac{\theta}{2}\right)\langle 1|+\rangle - e^{i\varepsilon_+ t}\sin\left(\frac{\theta}{2}\right)\langle 1|-\rangle \tag{40}$$

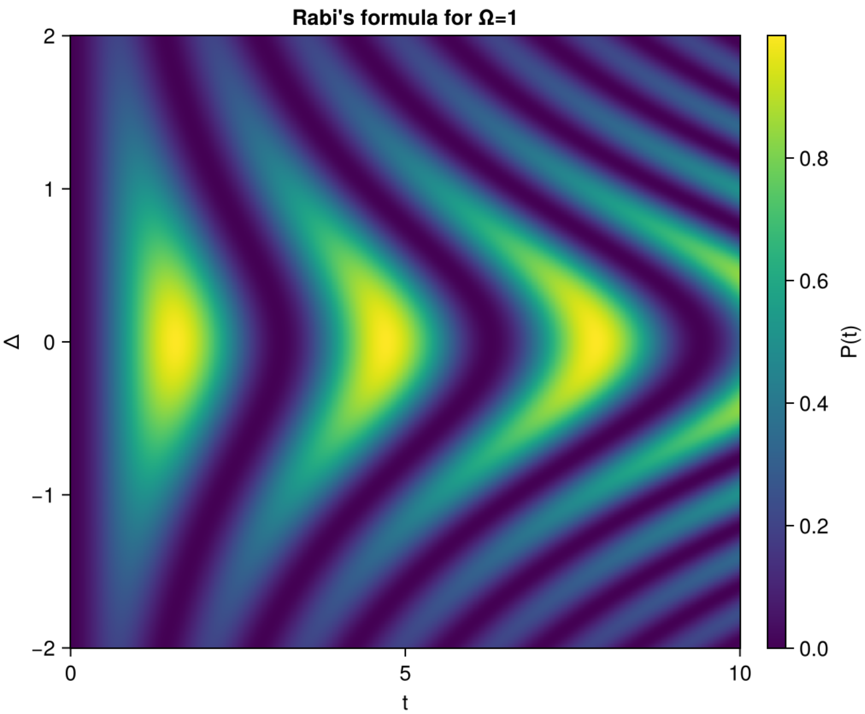
$$= e^{-i\varepsilon_+ t}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - e^{i\varepsilon_+ t}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \tag{41}$$

$$= i\sin(\theta)\sin(\varepsilon_+ t). \tag{42}$$

The probability of the system to be in the state $|1\rangle$ reads

$$|\langle 1|\psi(t)\rangle|^2 = \sin^2(\theta)\sin^2(\varepsilon_+ t) = \frac{\Omega^2}{\Delta^2 + \Omega^2}\sin^2\left(\sqrt{\Omega^2 + \Delta^2}t\right). \tag{43}$$

This is the famous Rabi's formula.



4. We finally compute the time average of the probability as

$$\bar{P}_1 = \lim_{t\to\infty} \frac{1}{T}\int_0^T P_1(t)dt = \frac{1}{2}\frac{\Omega^2}{\Delta^2+\Omega^2}, \tag{44}$$

   since the time average of the function $\sin^2(\omega t)$ is $1/2$.

5. We can use Julia to plot the Rabi frequency as a function of time t and detuning Δ for a specific value of the drive amplitude Ω. We provide the code here below:

```
using CairoMakie

Omega = 1
t = range(0, 10, 1000)
Delta = range(-2, 2, 1000)

P_rabi =  (Omega^2 ./(Delta'.^2 .+ Omega^2)) .* sin.(sqrt.(Omega^2 .+ Delta'.^2) .* t).^2

fig = Figure(size = (600, 500))
ax1 = Axis(fig[1, 1], title="Rabi's formula for Omega=1", xlabel="t", ylabel="Delta")
hm  = heatmap!(ax1, t, Delta, P_rabi, colormap=:viridis)
cb  = Colorbar(fig[1, 2], hm, label="P(t)")

fig
```