Solid state systems for quantum information, Session 9

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Exercise 1 : Jaynes-Cumming model: spectrum, entanglement and dynamics

In this exercise we are going to study the fundamental properties of the Jaynes-Cumming model, whose Hamiltonian in the rotating wave approximation is written as follows ($\hbar = 1$)

$$\hat{H} = \frac{\omega_q}{2}\hat{\sigma}^z + \omega_r \hat{a}^\dagger \hat{a} + g(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+). \tag{1}$$

The Jaynes-Cumming model describes the interaction between a quantized mode of the electromagnetic field and a two-level atom, and it is one of the paradigmatic models in quantum optics. In Eq. (9), ω_q is the qubit frequency, ω_r is the resonator frequency, and g is the light-matter coupling. The frequency distance between the two-level system and the resonator is called detuning, $\Delta = \omega_q - \omega_r$. We are going to study this model systematically, from its fundamental symmetries to the exact spectrum and the dynamics.

1. Determine the symmetries of the Hamiltonian. The eigenvectors of the non-interacting Hamiltonian ($g = 0$) are tensor products between the $\hat{\sigma}^z$ ground and excited states, and the Fock states, $\mathcal{H} = \text{span}\{|g, e\rangle \otimes |n\rangle_{n\in\mathbb{N}}\}$. These states are also called the bare eigenvectors. We will consider this basis throughout the exercise. Find the symmetry of $\hat{H}$ and the Hermitian operator $\hat{O}$ such that $[\hat{H}, \hat{O}] = 0$ (*i.e.*, $\hat{O}$ is a conserved quantity of the system). Write $\hat{O}$ in the eigenbasis of the non-interacting problem and in terms of the bosonic creation and annihilation operators and of $\hat{\sigma}^z$.

2. Find the exact spectrum of the Jaynes-Cumming model. To do this:

   (a) Compute the matrix elements of the interacting term (proportional to g) and show that $\hat{H}$ is block-diagonal with 2×2 blocks. Interpret this finding in light of what you found in the previous point.

   (b) Diagonalize the 2×2 block finding eigenvalues and eigenvectors of $\hat{H}$. Write the eigenvectors in terms of the bare eigenvectors. These eigenvectors are called dressed eigenvectors. Hint: To find the eigenvectors, use the rotation matrix (also called Bogoliubov matrix)

   $$U = \begin{pmatrix} \sin(\theta_n/2) & \cos(\theta_n/2) \\ \cos(\theta_n/2) & -\sin(\theta_n/2) \end{pmatrix} \tag{2}$$

   And compute the angle θ_n.

3. Show that the dressed states are generically entangled. Moreover, show that $|\Delta| = 0$ corresponds to maximal entanglement whereas $|\Delta| \to \infty$ gives back a product state. Provide an intuitive explanation about this finding.
   Hint: To prove that a quantum state $|\Psi\rangle$ of a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is entangled in

some basis which is the tensor product of two single-particle bases, $|\phi\rangle = |a\rangle \otimes |b\rangle$, you need to show that the reduced density matrix describing one of the two subsystem is not pure. To compute the reduced density matrix, you need to perform the partial trace, which amounts to tracing out one of the two subsystems:

$$\hat{\rho}_a = \mathrm{Tr}_b(\hat{\rho}) = \sum_b \langle b|\, \Psi\rangle\langle\Psi\, |b\rangle\,. \tag{3}$$

To asses whether $\hat{\rho}_a$ is pure or not, you can compute the purity $\gamma_a = \mathrm{Tr}(\hat{\rho}_a^2)$. If $\gamma_a < 1$ then the state $|\Psi\rangle$ is entangled. If $\gamma = 1/\dim(\mathcal{H}_a)$, then the system is maximally entangled.

4. We now analyze the Jaynes-Cumming dynamics. To solve for the dynamics, proceed with the following steps:

   (a) Write a single 2×2 block of the Hamiltonian $\hat{H}$ in terms of the Pauli matrices

   (b) Rewrite the time evolution operator in the form $\exp\left[i\frac{\Omega_n t}{2}(n_x\hat{\sigma}^x + n_z\hat{\sigma}^z)\right]$, with $n_x^2+n_z^2 = 1$. Calculate Ω_n, n_z and n_x. How is Ω_n called?

   (c) Rewrite the above results in terms of a 2×2 matrix.

   (d) Starting with the state $|\psi(0)\rangle = |g, n+1\rangle$ calculate the state $|\psi(t)\rangle$ at time t. Why is it sufficient to consider the time evolution of small blocks, instead of the time evolution of the full $\hat{H}$?

5. Finally, we study atomic inversion in the Jaynes-Cumming model. Use your results from the previous question to calculate the atomic inversion, given as $w(t) = |C_e(t)|^2 - |C_g(t)|^2$ when starting in the state $|\Psi(0)\rangle = |g, n+1\rangle$ with $C_g(t) = \langle g, n+1|\Psi(t)\rangle$ and $C_e(t) = \langle e, n|\Psi(t)\rangle$. What is the atomic inversion, if the detuning is $\Delta = 0$?

Exercise 2 : Schrieffer-Wolff transformation for transmon readout

In this exercise we are going to apply degenerate perturbation theory to describe the transmon qubit readout. The method we adopt is the Schrieffer-Wolff transformation. Consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{4}$$

where $\hat{H}_0$ is some unperturbed Hamiltonian of which we know the spectrum, $\hat{H}_0 |\Psi_n\rangle = E_n |\Psi_n\rangle$, $\hat{V}$ is an off-diagonal perturbation. The idea of the Schrieffer-Wolff transformation comes from the Baker-Campbell-Haussdorff expansion: we perform a unitary transformation on $\hat{H}$ with generator $\hat{S}$. Up to second order, we have

$$e^{\hat{S}}\hat{H}e^{-\hat{S}} = \hat{H} + [\hat{S}, \hat{H}] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{H}]]. \tag{5}$$

Now if we write $\hat{H} = \hat{H}_0 + \hat{V}$ we obtain

$$e^{\hat{S}}\hat{H}e^{-\hat{S}} = \hat{H}_0 + \hat{V} + [\hat{S}, \hat{H}_0] + [\hat{S}, \hat{V}] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{H}_0]] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{V}]]. \tag{6}$$

We now impose that the generator $\hat{S}$ is such that $\hat{V} = -[\hat{S}, \hat{H}_0]$, *i.e.*, it cancels the contribution of $\hat{V}$ at the first order. This leads to the second-order Schrieffer-Wolff formula

$$\hat{H}_{\text{eff}} = e^{\hat{S}}\hat{H}e^{-\hat{S}} \simeq \hat{H}_0 + \frac{1}{2}[\hat{S}, \hat{V}]. \tag{7}$$

The problem is of course finding $\hat{S}$. We state here, without proving it, that the Schrieffer-Wolff generator at first order is given by

$$\hat{S} = \sum_{n,m} \frac{\langle\Psi_n| \hat{V} |\Psi_m\rangle}{E_n - E_m} |\Psi_n\rangle \langle\Psi_m| . \tag{8}$$

The above two equations provide all the ingredients to compute low-energy effective Hamiltonians. We apply this formalism to two examples, both relevant for circuit QED.

1. Jaynes-Cumming model. Consider the Hamiltonian

$$\hat{H}_{\text{JC}} = \frac{\omega_q}{2}\hat{\sigma}^z + \omega_r \hat{a}^\dagger \hat{a} - g(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+). \tag{9}$$

   Suppose $g \ll |\Delta| = |\omega_q - \omega_r|$. Compute the effective low-energy Hamiltonian by means of a second-order Schrieffer-Wolff transformation. Justify the use of perturbation theory in this context and give a physical interpretation about the terms appearing in the $\hat{H}_{\text{eff, JC}}$ you find.

2. Dispersive readout of a superconducting transmon qubit. In the previous point, we modeled the transmon qubit as a two-level system. We now want to go beyond this (very) simple approximation and we want to take into account the multilevel structure of the transmon. Consider the Hamiltonian

$$\hat{H}_{\text{cQED}} = \omega_r \hat{a}^\dagger \hat{a} + \omega_q \hat{b}^\dagger \hat{b} - \frac{E_c}{2}\hat{b}^{\dagger 2}\hat{b}^2 - g(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \tag{10}$$

where $\hat{a}$ and $\hat{b}$ are the bosonic modes of the resonator and of the transmon, respectively. Suppose $g \ll |\Delta| = |\omega_q - \omega_r|$. Compute the effective low-energy Hamiltonian $H_{\mathrm{eff,\,cQED}}$ by means of a second-order Schrieffer-Wolff transformation, and neglect possible counter-rotating terms.

3. Starting from $H_{\mathrm{eff,\,cQED}}$, truncate all the energy levels but the first two and compute the dispersive shift χ (half of the resonator's energy different when the qubit is up or down, respectively). Compare it to the Hamiltonian $\hat{H}_{\mathrm{eff,\,JC}}$ you found in the previous point. Why are they not coinciding?