Solid state systems for quantum information, Session 7

Assistants : franco.depalma@epfl.ch, filippo.ferrari@epfl.ch

1 Exercises

Exercise 1 : Driven linear and non-linear oscillator

The Cooper-pair box Hamiltonian

$$\hat{H}_{CPB} = 4E_C(\hat{n} - n_g)^2 - E_J \cos\left(\hat{\delta}\right), \tag{1}$$

in the so-called transmon limit, that is $E_J \gg E_C$, has low-lying levels with energy scale $\sqrt{8E_J E_C} \ll E_J$ much smaller than the amplitude of the cosine potential in the phase coordinate. Therefore, the phase δ is always close to zero when the system is in one of the low-lying energy levels, and we can perform Taylor expansion of the cosine as $-E_J \cos\left(\hat{\delta}\right) \approx \text{const} + E_J \frac{\hat{\delta}^2}{2} - E_J \frac{\hat{\delta}^4}{24}$. As the charge dispersion decreases exponentially with E_J/E_C, we neglect the charge offset n_g and reach the Hamiltonian

$$\hat{H}_T = 4E_C \hat{n}^2 + \frac{1}{2}E_J \hat{\delta}^2 - \frac{1}{24}E_J \hat{\delta}^4. \tag{2}$$

1. Treating the quartic term as a perturbation leads to a renormalized transition frequency ω, as well as a renormalized anharmonicity α. Show that the Hamiltonian in its second quantized form can be written as:

   $$\hat{H} = \hbar\left(\omega \hat{a}^\dagger \hat{a} + \frac{1}{2}\alpha \hat{a}^{\dagger 2} \hat{a}^2\right). \tag{3}$$

   What is the expression of the resonance frequency ω and the anharmonicity α?

2. We will take typical values for a transmon of $\omega/(2\pi) = 6$ GHz, and anharmonicity of $\alpha/(2\pi) = -300$ MHz. In order to demonstrate that such non-linear oscillator can be operated as an effective two-level system, i.e. a qubit, we study the time-evolution under the influence of a driving field. We model this drive as a pulse with a Gaussian envelope of the form $H_1 = \hbar A_0 e^{(-t/\tau)^2}\left(a e^{i\omega_d t} + a^\dagger e^{-i\omega_d t}\right)$, with $\tau = 5$ ns. The file Rabi_pulses-Questions.ipynb on Moodle provides a template on how to proceed.

   Define the Hamiltonian operator, and plot the evolution of the populations of the ground state and the first excited state over time, assuming an amplitude $A_0/(2\pi) \in \{0.18, 0.4\}$ GHz.

3. Plot the average number of excitations $\langle n \rangle = \left\langle a^\dagger a \right\rangle$ in the system at the end of the pulse versus the pulse amplitude. What do you observe?

   Hint: the qutip "mesolve" (master equation solver) function takes as parameters, in order, the Hamiltonian, the initial vector, the collapse operators (which we do not use in this exercise, thus pass "[]" as an argument), a list of operators for which to return the expectation value (if the list is empty, the function returns the the state vector), and additional parameters that one can sweep in the Hamiltonian.

4. Perform the same drive on a linear system, i.e. $\alpha = 0$. What qualitative change do you observe? Compare the final number of excitations in the two systems.

5. For a drive amplitude of $A_0/(2\pi) = 0.18$ GHz, plot the occupation probability in the Fock levels $0, 1, ...$ Discuss the state obtained in both cases.

Exercise 2 : Circuit QED Hamiltonians in First Quantization

In this computational exercise we are going to build up the main Hamiltonians seen in the lecture notes from a different point of view. We will focus on the quantized LC circuit, whose Hamiltonian is given by

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \tag{4}$$

being $\hat{Q}$ and $\hat{\Phi}$ the charge and flux operators respectively, and C and L the capacitance and the inductance of the circuit, respectively. We will also focus on the transmon qubit, whose Hamiltonian reads instead

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\phi} \tag{5}$$

where $\hat{n}$ and $\hat{\phi}$ are the charge number and phase operators, respectively, while E_C and E_J are the charging and Josephson energies, respectively.

1. We introduce the charge number and phase operators for the LC circuit

$$\hat{n} = \hat{Q}/2e, \qquad \hat{\phi} = (2\pi/\Phi_0)\hat{\Phi}, \tag{6}$$

   being e the electron charge and $\Phi_0 = h/2e$ the magnetic flux quantum. From now one, we assume $\hbar = e = 1$. We also introduce the charging and the inductive energies as $E_C = 1/2C$ and $E_L = 1/4L$. Rewrite Eq. (4) in terms of $\hat{n}$, $\hat{\phi}$, E_C and E_L.

2. Expand $\hat{n}$ and ϕ in terms of creation and annihilation operators

$$\hat{\phi} = \left(\frac{2E_C}{E_L}\right)^{1/4}(\hat{a} + \hat{a}^\dagger), \qquad \hat{n} = \frac{i}{2}\left(\frac{E_L}{2E_C}\right)^{1/4}(\hat{a}^\dagger - \hat{a}). \tag{7}$$

   Show that this choice diagonalizes the Hamiltonian. Write the eigenenergies in terms of E_C and E_L. This is the well known Second Quantization procedure.

3. We now compute eigenstates and eigenenrgies of Eq. (4) in First Quantization. Since number and phase operators are canonically conjugate variables, we can express $\hat{n}$ in the ϕ-basis as $\hat{n} = -i\partial/\partial\phi$. This allows us to write an Hamiltonian in the ϕ-basis only, $\mathcal{H} = T(\phi) + V(\phi)$. Using the central difference approximation scheme, write $\partial/\partial\phi$ and $\partial^2/\partial\phi^2$ as square matrices.

   Notice that the potential term $V(\phi)$ in the ϕ-basis is unbounded. Give an estimate on the maximal phase $\phi_{\max}$ needed to describe the first n quantized energy levels of the LC circuit.

4. Start from the following physical parameters for the LC oscillator:

$$\omega_r/2\pi = 9.375\,\text{GHz}, \qquad Z_r = 25\,\Omega. \tag{8}$$

   Take $N_\phi = 2000$ points in the interval $[-\phi_{\max}, \phi_{\max}]$, where to find $\phi_{\max}$ you suppose $n = 20$ energy levels.

   Build the matrix for the quantum LC circuit and diagonalize it numerically, obtaining the eigenenergies E_n and the eigenstates $\Psi_n(\phi)$. Plot the potential $V(\phi)$, the first $n = 10$ energy levels ε_n obtained from the Second Quantization procedure, and the first $n = 10$ rescaled wave functions $|\tilde{\Psi}_n(\phi)|^2 = A|\Psi_n(\phi)|^2 + E_n$ (being E_n the energies in First Quantization). Take $A = 10^4$ (Since the amplitude of the wave function is very small, you need to amplify them just to visualize them on the plot).

5. We now focus on the transmon qubit in First Quantization. Expand $\hat{n}$ and $\hat{\phi}$ in terms of creation and annihilation operators

$$\hat{\phi} = \left(\frac{2E_C}{E_J}\right)^{1/4} (\hat{b} + \hat{b}^\dagger), \qquad \hat{n} = \frac{i}{2}\left(\frac{E_J}{2E_C}\right)^{1/4} (\hat{b}^\dagger - \hat{b}). \tag{9}$$

Repeat the procedure of the first exercise and write down the Second Quantized form of Eq. (5), upon expanding the cosine potential up to the fourth order in $\hat{\phi}$.

6. First Quantization is particularly advantageous for nonlinear circuit QED system because the nonlinear potential are tipically diagonal in the ϕ-basis. Using the same procedure of Point 3, write the transmon Hamiltonian in the ϕ-basis as $\mathcal{H} = T(\phi) + V(\phi)$ (use again the central difference approximation scheme).

   Unlike the harmonic oscillator, the potential $V(\phi)$ is now periodic in ϕ and we can restrict the phase in the interval $[-\pi, \pi]$. Start from the following physical parameters for the transmon qubit:

$$E_C/2\pi = 300\,\text{MHz}, \qquad E_J = 50 \times E_C. \tag{10}$$

   Take $N_\phi = 2000$ points in the interval $[-\pi, \pi]$.

   Build the matrix for the transmon qubit and diagonalize it numerically, obtaining the eigenenergies E_n and the eigenstates $\Psi_n(\phi)$. Plot the potential $V(\phi)$, the first $n = 9$ energy levels obtained from the Second Quantized Hamiltonian plus the First Quantized ground state energy, $\varepsilon_n + E_0$ (this fixed a common zero-point energy), and the first $n = 9$ rescaled wave functions $|\tilde{\Psi}_n(\phi)|^2 = A|\Psi_n(\phi)|^2 + E_n$ (being E_n the energies in First Quantization). Take $A = 10^4$ as in Point 4.

   Discuss the discrepancies between the First and the Second Quantized model.

Exercise 3 : Ramsey interferometry

The method of Ramsey interferometry is a widely used method in atomic clocks, modern atom interferometers and quantum logic gates. It allows precise measurement of the phase of a quantum state. We can illustrate the procedure for Ramsey interferometry using the following circuit diagram:

$$|0\rangle \; — \boxed{R_x(\pi/2)} — \boxed{U_1} — \boxed{R_x(\pi/2)} — \; |\psi(T)\rangle$$

We prepare the qubit in state $|0\rangle$ and then rotate it into the superposition of $|0\rangle$ and $|1\rangle$ states using the gate $R_x(\pi/2)$, where $R_x(\theta) = e^{-i\theta\sigma_x/2}$, rotates the qubit by the angle θ around the x axis of the Bloch sphere (and similarly for $R_z(\theta) = e^{-i\theta\sigma_z/2}$). The qubit is then allowed to evolve freely for time T, and this is shown as gate U_1. In our case, the qubit is placed in a (time-varying) magnetic field $B(t)$, such that the energies of the two qubit states are split by $g\mu_B B(t)$, with g the electron g-factor, μ_B the Bohr magneton. Setting the energy of the lower state to 0, the Hamiltonian that governs the qubit's free evolution is given by $H = g\mu_B B(t)\,|1\rangle\langle 1|$. Given this, U_1 corresponds to $R_z\left(\int_0^T \frac{g\mu_B}{\hbar} B(t')\right)$. After the free evolution the qubit is rotated again using $R_x(\pi/2)$ to the state $|\psi(T)\rangle$ and is measured in the computational basis.

1. Draw the state after each step of the circuit diagram on a Bloch sphere. Do this for $U_1 = R_z(0)$ and $U_1 = R_z(\pi)$. Can you see the oscillation of $P_0(T) = |\langle 0|\psi(T)\rangle|^2$ with varying θ in $U_1 = R_z(\theta)$?

2. Calculate the probability of finding the qubit in state $|0\rangle$: $P_0(T) = |\langle 0|\psi(T)\rangle|^2$.
   Hint: $\exp\left(i\theta\sigma_{x/z}\right) = \mathbb{1}\cos\theta + i\sigma_{x/z}\sin\theta$.

3. We now want to investigate the effect of a noisy classical magnetic field. We will first assume $B(t)$ is constant for the duration of the experiment but upon repetition of the experiment takes on different, random values. We then can calculate the expectation value of observables over the ensemble average of magnetic field values. Assume $B(t) = B$ where B is a time-independent Gaussian random variable, with mean B_0 and variance B_1^2. Find $\langle P_0(T)\rangle$, where the brackets $\langle\ \rangle$ indicate the expectation value of the random variable.
   Hint: Recall that a Gaussian of a random variable X with mean μ and variance σ^2 is given by $\langle e^{-iXt}\rangle = e^{-i\mu t}e^{-\sigma^2 t^2/2}$.