Solid state systems for quantum information, Session 4

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1 Exercises

Exercise 1 : The Cooper pair box (CPB) Hamiltonian and the transmon limit

From the second exercise sheet and from the lecture, you should have seen that the CPB Hamiltonian reads as:

$$\hat{H}_{CPB} = 4E_C\left(\hat{N} - n_g\right)^2 - E_J\cos\left(\hat{\delta}\right),$$

where $\hat{N}$ is the charge operator, $\hat{\delta}$ the phase operator, E_C the charging energy, E_J the Josephson energy and n_g an offset charge. $\hat{N}$ and $\hat{\delta}$ are conjugate operators, meaning that their commutator is $[\hat{\delta}, \hat{N}] = i$.

1. Show that $[\hat{N}, e^{i\hat{\delta}}] = e^{i\hat{\delta}}$.
   Hint: Use a power series to express the exponential.

2. Show that $e^{i\hat{\delta}}\,|m\rangle = |m+1\rangle$, where $|m\rangle$ is an eigenstate of the charge operator, i.e. $\hat{N}\,|m\rangle = m\,|m\rangle$.
   Hint: Make use of the identity shown in 1.

3. Use the identities from 1. and 2. to show that the CPB Hamiltonian in the charge basis reads

$$\hat{H}_{CPB} = \sum_{m=-\infty}^{+\infty}\left[4E_C(m-n_g)^2\,|m\rangle\langle m| - \frac{1}{2}E_J(|m+1\rangle\langle m| + |m\rangle\langle m+1|)\right] \tag{1}$$

4. Numerically diagonalize the CPB Hamiltonian for various ratios of $E_J/E_C \in \{1, 5, 10, 30\}$ using qutip. For this purpose, truncate the Hilbert space at a suitable dimension corresponding to a maximum charge $|m_{max}|$.

   - Plot the eigenenergies $E_g(n_g)$, $E_e(n_g)$, $E_f(n_g)$, corresponding to the three lowest-lying eigenstates $|g\rangle$, $|e\rangle$, $|f\rangle$ of the CPB Hamiltonian as a function of n_g.
   - For $n_g = 0$, plot the probability $p_m = |\langle m|i\rangle|^2$ to be in charge state $|m\rangle$ for the two eigenstates $|i\rangle \in |g\rangle, |e\rangle$. Convince yourself that you have chosen a sufficiently large $|m_{max}|$ in your simulations.

5. From your numerical result from 4., extract the charge dispersion $\varepsilon = E_{ge}(n_g = 0) - E_{ge}(n_g = 1/2)$ of the $|g\rangle \leftrightarrow |e\rangle$ transition, where $E_{ge} = E_e - E_g$, for the four values of E_J/E_C, and plot them as function of E_J/E_C.

6. Show that the charge operator has the following representation in the phase eigenbasis $\{< |\delta\rangle\}$:

$$\langle\delta|\, \hat{N}\, |\Psi\rangle = i\frac{\partial}{\partial\delta}\Psi(\delta), \tag{2}$$

where $\langle\delta|\Psi\rangle = \Psi(\delta)$ is the wave function in phase space. Make use of the charge basis representation of the phase eigenstates, $|\delta\rangle = \sum_{m=-\infty}^{+\infty} e^{im\delta}\, |m\rangle$.