Solid state systems for quantum information, Session 3

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1 Exercises

Exercise 1 : Capacitance of a coplanar plate capacitor

Superconducting circuits are based on inductive and capacitive elements patterned into a superconducting thin film, which resides on a dielectric substrate. Here, we estimate the capacitance and the dipole moment of a coplanar, parallel plate capacitor made of two rectangular pads, see Fig. 1. By combining such capacitors with inductive elements, one can e.g. build on-chip LC resonators.

1. Estimate the order of magnitude of the capacitance between the two pads of this element. Apply this to typical values $a = 300$ μm, $b = 400$ μm, $l = 100$ μm, film thickness $t = 150$ nm, and relative dielectric constant $\varepsilon_r \propto 10$. Assume for simplicity that the electric field lines extend mainly into the substrate and that the substrate thickness is large compared to the one of the capacitor.

2. Estimate the inductance required to obtain a lumped element resonator with a resonance frequency of $f \simeq 6$ GHz?

3. Estimate the dipole moment of this capacitor for a single Cooper pair located on one of the islands. The dipole operator in this example is given by $d = lQ$ with l in this case being the separation between the two pads and Q is the charge.

4. Calculate the charge zero point fluctuation, defined as $Q_{ZPF} = \sqrt{\frac{\hbar}{2Z}}$, where $Z = \sqrt{\frac{L}{C}}$. How does it compare to a single Cooper pair?

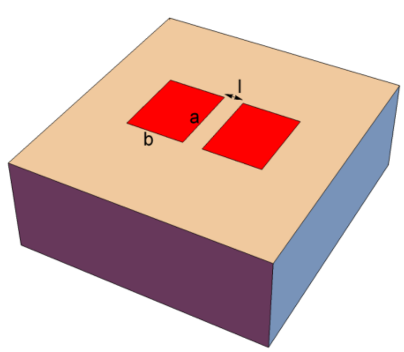

Figure 1: Sketch of a capacitor formed by two pads of a superconductor (red) on an insulating substrate (grey).

Exercise 2 : Input-output theory

We consider an LC-resonator which is capacitively coupled with a rate of κ_{ext} through capacitor C_c to a semi-infinite transmission line with a characteristic impedance of $Z_0 = 50\ \Omega$. The LC-resonator has an internal loss rate of κ_{int} and is depicted in Fig. 2. We consider a classical coherent input field $\langle \hat{b}_{\text{int}}(t) \rangle = \beta_{in} e^{-i\omega t}$ traveling towards the LC-resonator, as well as an output field $\langle \hat{b}_{\text{out}}(t) \rangle$ being reflected from it. The equation of motion, as a function of time, for the system operator $\hat{a} = \hat{a}(t)$ reads

$$\frac{\partial \hat{a}}{\partial t} = \frac{i}{\hbar}\left[\hat{H}_{\text{sys}}, \hat{a}\right] - \frac{\kappa_{\text{int}} + \kappa_{\text{ext}}}{2}\hat{a} + \sqrt{\kappa_{\text{ext}}}\hat{b}_{in} \tag{1}$$

with $\hat{H}_{\text{sys}} = \hbar\omega_0 \hat{a}^\dagger \hat{a}$ being the system Hamiltonian of the LC-resonator. The boundary condition which relates the output to the input field is

$$\hat{b}_{\text{in}}(t) + \hat{b}_{\text{out}}(t) = \sqrt{\kappa_{\text{ext}}}\hat{a}(t) \tag{2}$$

1. Calculate the commutator $\left[\hat{H}_{\text{sys}}, \hat{a}\right]$ and use the Ansatz $\langle \hat{a}(t) \rangle = \alpha e^{-i\omega t}$ for the time dependence of the expectation value of the system operator and simplify the equation of motion. Express the amplitude α of the expectation value of the system operator as a function of the angular frequency ω and the amplitude of the input field β_{in}.

2. Solve for the frequency dependent reflection coefficient $S_{11}(\omega) = \frac{\langle \hat{b}_{\text{out}} \rangle}{\langle \hat{b}_{\text{in}} \rangle}$.

3. Plot the squared absolute value, and the real and imaginary parts of the reflection coefficient $S_{11}(\omega)$ for a resonator with parameters $\omega_0/2\pi = 6$ GHz and $\kappa_{\text{ext}}/2\pi = 5$ MHz. Use three different internal loss rates rates of $\kappa_{\text{int}}/2\pi \in \{0, 5, 100\}$ kHz. What changes when the internal loss rate is increased? Discuss all three cases.

   We now consider an LC-resonator which is capacitively coupled to two semi-infinite transmission lines, as shown Fig. 3.

4. Calculate the transmission coefficient $S_{21}(\omega) = \frac{\langle \hat{b}_{2,\text{out}} \rangle}{\langle \hat{b}_{1,\text{in}} \rangle}$. Show that the transmission coefficient $S_{21}(\omega = \omega_0) \to 1$ for a lossless LC-resonator.

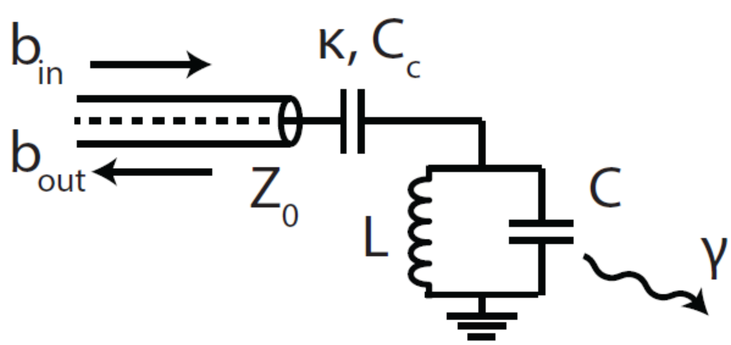

Figure 2: Electrical circuit of an LC-resonator capacitively coupled to a semi-infinite transmission line for a measurement of the reflection coefficient S_{11}.

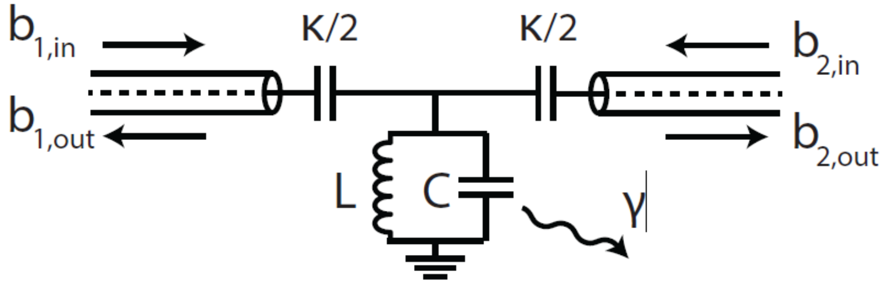

Figure 3: Electrical circuit of an LC-resonator capacitively coupled to two semi-infinite transmission lines for a measurement of the transmission coefficient S_{21}.

5. Plot the squared absolute value, and the real and imaginary parts of the transmission coefficient $S_{21}(\omega)$ for the same parameter values as in 3. In what way do the results differ from those for the reflection coefficient in 3.

Exercise 3 : Coupled-cavity Arrays

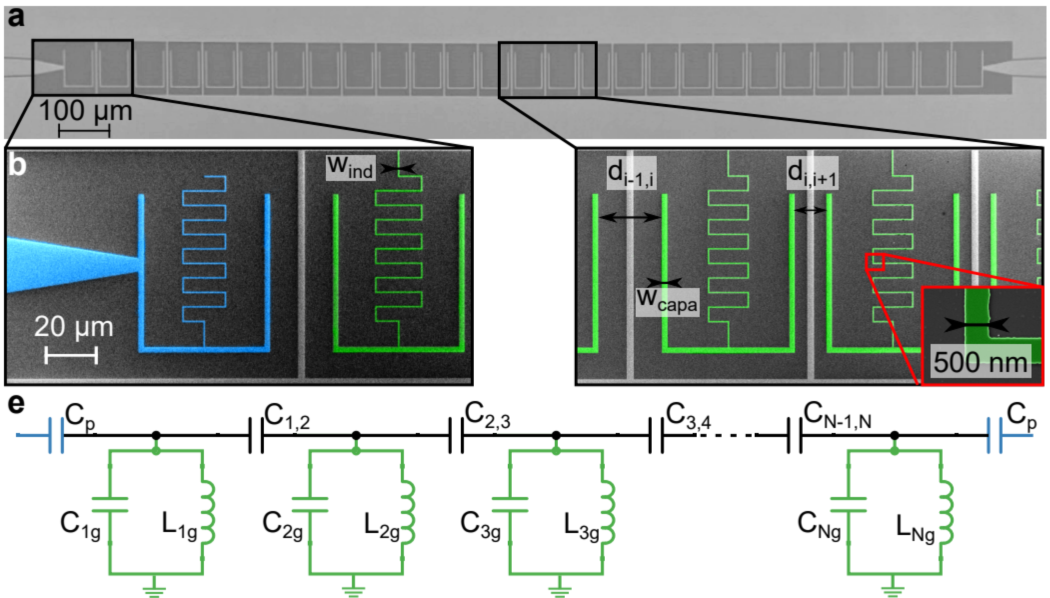

Figure 4: Coupled cavity array made of a chain of coupled LC resonators. The figure is taken from https://arxiv.org/abs/2403.18150.

In this exercise we study the transmission coefficients of coupled cavity arrays. We consider the system described in Fig. 4, which is chain of coupled L LC resonators.

1. Quantize the circuit and write down the equivalent Hamiltonian in terms of bosonic creation and annihilation operators $\hat{a}_\ell$ and $\hat{a}_\ell^\dagger$.

2. Assume periodic boundary conditions and diagonalize the Hamiltonian by passing in Fourier space by means of the momentun representation of the creation and annihilation operators

$$\hat{a}_\ell = \frac{1}{\sqrt{L}} \sum_k \hat{a}_k e^{ikx_\ell}, \qquad \hat{a}_\ell^\dagger = \frac{1}{\sqrt{L}} \sum_k \hat{a}_k^\dagger e^{-ikx_\ell}. \tag{3}$$

3. Suppose that now all the resonators have an intrinsic decay rate equal to κ_{int}, and that the first and the last resonators, due to the coupling with external feedlines, have an additional external decay rate equal to κ_{ext}. Write down the input-output equations for the intracavity field $\langle \hat{a}_\ell \rangle$.

4. Re-write the input-output relations in the compact form

$$\frac{\partial}{\partial t} \langle \vec{\hat{a}} \rangle = i\mathcal{M} \langle \vec{\hat{a}} \rangle, \tag{4}$$

where $\mathcal{M}$ is a $L \times L$ non-Hermitian matrix. Show that it is possible to diagonalize $\mathcal{M}$ with a matrix transformation, obtaining eigenmodes and eigenfrequencies of the system. Numerically diagonalize $\mathcal{M}$ for chains with $L = 10$ and $L = 50$ resonators. Discuss the physical interpretation of the system's eigenfrequencies, in light of what you found in point 2.

5. We now want to explore the response of the system to the injection of external photons, that we model with an external classical drive on the first cavity, $\langle \hat{b}_{\text{in}}(t) \rangle = \beta_{\text{in}} e^{i\omega t}$. Modify the input-output equations to account for the presence of the external drive. Derive the transmission coefficient $S_{1,L}(\omega)$ in terms of the eigenvectors and eigenvalues of $\mathcal{M}$. Plot $S_{1,L}(\omega)$ as a function of the drive frequency ω for chains of $L = 10$ and $L = 50$ resonators. Use the following parameters: $\omega_0/2\pi = 6$ GHz, $\kappa_{\text{int}}/2\pi = 100$ kHz, $\kappa_{\text{ext}}/2\pi = 10$ MHz, $g/2\pi = 200$ MHz.

Exercise 4 : Superconducting loops

In the superconducting state of a metal, the electrons close to the Fermi energy pair up with opposite spin and momentum to form Cooper pairs. All the Cooper pairs will have the same energy and they realize a macroscopic quantum state across the entire piece of the superconductor. Because this superconducting condensate is a coherent quantum state, it can be described by a wavefunction, which takes the form of

$$\psi(\mathbf{r}) = \sqrt{n_s} e^{i\varphi(\mathbf{r})} \tag{5}$$

where $\boldsymbol{r}$ is the position inside the superconductor, n_s is the density of the superconducting Cooper pairs and $\varphi(\boldsymbol{r})$ is the phase of the wavefunction. While inside the superconductor the amplitude of the wavefunction $\sqrt{n_s}$ is constant, the phase $\varphi(\boldsymbol{r})$ can change as a function of position. In this problem, we investigate how the phase behaves in a cylindrical superconductor in magnetic field.

1. In class, we saw that the change in phase with a current flowing in a superconductor from a point X to Y was defined as:

$$\varphi_{XY} = \frac{1}{\hbar} \int_X^Y p(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{6}$$

where $p = 2m\boldsymbol{v} + 2e\boldsymbol{A}$ in a magnetic field $\boldsymbol{B}$, with $\boldsymbol{A}$ being the vector potential respecting the relation $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. We consider a thin cylindrical superconductor with radius R and long axis oriented along the z direction (see Fig. 5). The external magnetic field also points along the z axis, and has a magnitude of B_0 such that $\boldsymbol{B} = (0, 0, B_0)$ in the Cartesian $x - y - z$ coordinate system. Importantly, the choice of the vector potential $\boldsymbol{A}$ is not unique; as long as it satisfies that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the vector potential describes correctly the system. Consequently, the phase of the superconductor is also not unique, which is not surprising because the phase of a wavefunction is not an observable quantity. In general, the vector potential can be changed by a gauge transformation such that

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi, \tag{7}$$

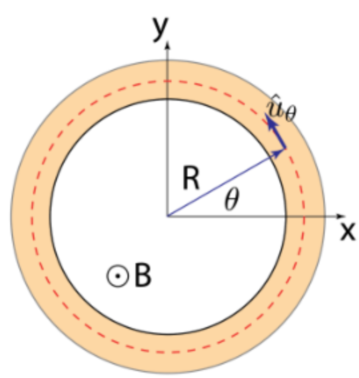

Figure 5: Top view of the cylindrical superconductor.

where χ is a scalar function. Here, we consider the Coulomb gauge, i.e. $\nabla\chi = 0$, so that $\boldsymbol{A} = (-yB_0/2, xB_0/2, 0)$.

Knowing that the current density $\boldsymbol{J}$ is given by $\boldsymbol{J} = 2en_s\boldsymbol{v}$, write the phase difference as a function of $\boldsymbol{J}$ and $\boldsymbol{A}$. Then show that the magnetic flux through the loop is quantized, by integrating over a closed loop and using the fact that the superconducting phase is periodic.

2. Now, let us consider a superconducting loop that contains two Josephson junctions (JJs), as depicted in Fig. 6, which is called a superconducting quantum interference device (SQUID). The current through a JJ can be described by the first Josephson equation:

$$I(\delta) = I_0 \sin(\delta), \tag{8}$$

where I_0 is the critical current of the junction, which depends on its geometry, and δ is the phase difference between the superconductors before and after the junction.

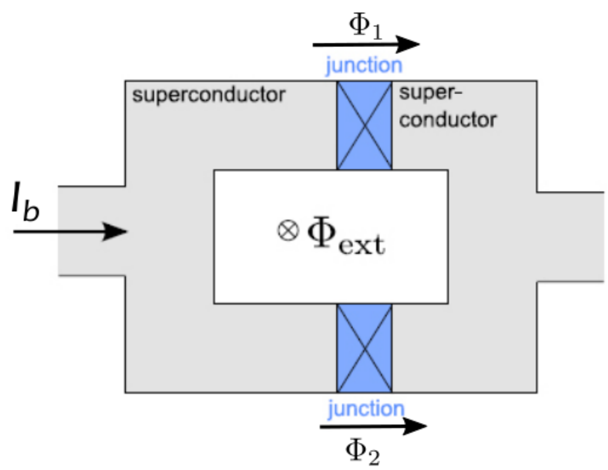

Figure 6: SQUID loop.

(a) Assuming that both JJs have the same critical current, what is the total current through the SQUID as a function of the phase differences δ_1 and δ_2 across JJs 1 and 2, respectively?

(b) Now we want to investigate the phase across the loop. In question 1., the gauge choice for $\boldsymbol{A}$ did not matter, since the contribution of χ vanishes when integrating over a closed loop. Since we now have two JJs in the loop, we cannot take a closed loop integral through the superconductor anymore. Therefore, we have to redefine the phase difference δ such that it becomes gauge invariant:

$$\delta = \varphi_Y - \varphi_X - \frac{2\pi}{\Phi_0}\int_X^Y \boldsymbol{A}(\boldsymbol{r})d\boldsymbol{r}, \tag{9}$$

where φ_X and φ_Y are the superconducting phases before and after the JJ, respectively. Calculate the phase change across the SQUID by splitting the integral into four parts, two parts through the superconductor and one part across each JJ.

(c) Using the result from a) and b), write the current total current through the SQUID as a function of the magnetic flux through the loop. Discuss how this result can be used to tune SQUIDs in a real device.