Solid state systems for quantum information, Session 2

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1 Exercises

Exercise 1 : Coupled LC resonators

Consider two LC resonators, see Fig. 1, with respective inductance, capacitance values L_1, C_1 and L_2, C_2. These resonators are capacitively coupled through a capacitance C_0. The flux variable at the i-th independent node corresponds to ϕ_i.

1. Write down the Lagrangian $\mathcal{L}(\phi, \dot{\phi})$ of the system as a quadratic function of the node flux variables ϕ_i and their derivatives $\dot{\phi}_i$. Introduce the capacitance matrix $\mathbb{C}$ and the inverse of the inductance matrix $\mathbb{L}^{-1}$ and use the flux variables and their derivatives in the vector representation,

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \text{and} \qquad \dot{\vec{\phi}} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

2. Perform the Legendre transformation analytically and extract the Hamiltonian H, as a function of the charge $Q_i = \partial\mathcal{L}(\phi, \dot{\phi})/\partial\dot{\phi}_i$ and the flux variables ϕ_i. Rewrite this Hamiltonian as a quadratic form, using $\mathbb{C}^{-1}$ and $\mathbb{L}^{-1}$.

   Hint: A square 2×2 matrix is inverted by

$$\mathbb{A}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{\det(\mathbb{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

3. To perform quantization, first rewrite the Hamiltonian $\mathcal{H}$ in terms of both inductances L_1, L_2 and the bare (uncoupled) angular frequencies ω_i with $\omega_i^2 = \mathbb{L}_{ii}^{-1}(\mathbb{C}^{-1})_{ii}$. Summarize the capacitive coupling in a single constant $\beta = C_0/\sqrt{(C_1 + C_0)(C_2 + C_0)}$. Perform the quantization by introducing the corresponding quantum operators $\hat{Q}_i$ and $\hat{\phi}_i$ which satisfy the canonical commutation relation $[\hat{Q}_i, \hat{\phi}_j] = -i\hbar\delta_{ij}$. Subsequently, use the following definition of the charge and flux operator to write the Hamiltonian H in terms of annihilation and creation operators, $\hat{a}_i$ and $\hat{a}_i^\dagger$,

$$\hat{Q}_i = -i\sqrt{\frac{\hbar}{2L_i\omega_i}}(\hat{a}_i - \hat{a}_i^\dagger), \qquad \text{and} \qquad \hat{\phi}_i = \sqrt{\frac{\hbar L_i \omega_i}{2}}(\hat{a}_i + \hat{a}_i^\dagger)$$

4. Apply the rotating wave approximation (RWA) on the coupling term and diagonalize the resulting quadratic Hamiltonian with a Bogoliubov transformation. Discuss the physical interpretation of the obtained eigenenergies and eigenmodes.

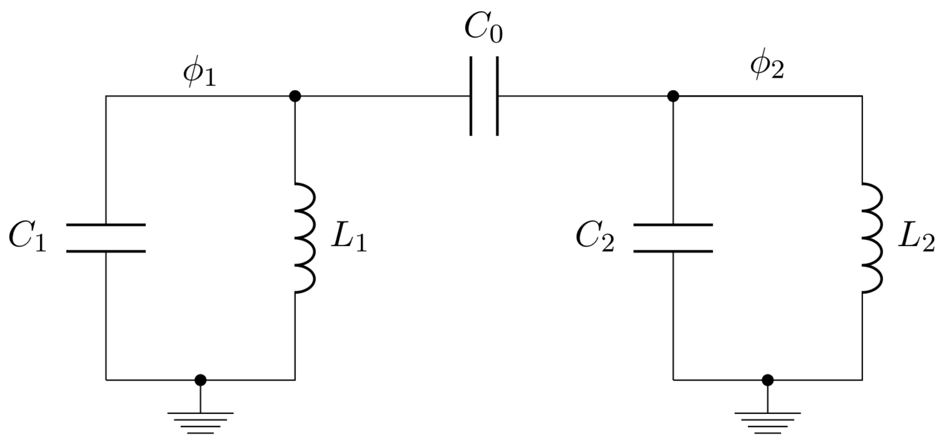

Figure 1: Circuit diagram of two capacitively coupled LC resonators.

<u>Hint:</u> To apply the rotating wave approximation, *suppose* that the Hamiltonian is driven at a frequency ω_d. Perform a unitary transformation in the frame rotating at the pump frequency and understand which terms are fastly oscillating in time. To diagonalize the RWA Hamiltonian, consider a Bogoliubov transformation, which consists in the following ansatz

$$\hat{\alpha} = u\hat{a}_1 + v\hat{a}_2, \tag{1}$$

where u and v are generally complex amplitudes. To find u and v impose $[\hat{H}, \hat{\alpha}] = -E\hat{\alpha}$ and solve the corresponding 2×2 system, and remember that $[\hat{\alpha}, \hat{\alpha}^\dagger] = \mathbb{1}$.

5. Consider now the following values for the capacitances of the two LC oscillators: $C_1 = C_2 = 70\,\text{fF}$, and the following value for the inductance $L_1 = 10\,\text{nH}$. Compute

   - The bare mode frequencies ω_1 and ω_2 as a function of L_2.
   - The coupled mode frequencies of the RWA Hamiltonian according to the formula you have obtained in the previous point, supposing $C_0 = 10\,\text{fF}$.

   Discuss your findings from a physical point of view.

Exercise 2 : Circuit Quantization

1. The goal of this exercise is to find the Lagrangian and quantized Hamiltonian of a LC resonator capacitively coupled to a time variable voltage source (see Fig. 2a).

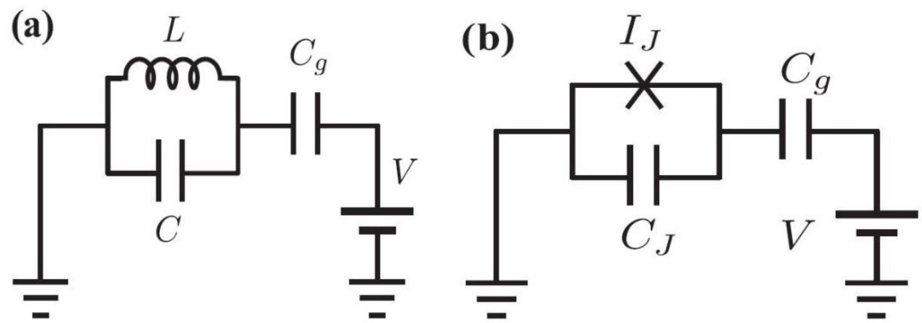

Figure 2: a) Equivalent circuit for an LC resonator consisting on an inductor in parallel with a capacitor, subject to an external potential V(t). b) Equivalent circuit for a non-linear inductor (a Josephson junction) in parallel with a capacitor, subject to an external potential V(t).

   (a) First, decompose the circuit to identify the branch and node fluxes. Find the relation between them. You should end up with only one flux variable.
   (b) Find the Lagrangian of the system.
   (c) With the Legendre transformation, find the associated quantized Hamiltonian.

2. The circuit in Fig. 2b models a Cooper Pair Box or equivalently, as you will see in future sessions, a transmon qubit. It mirrors the LC resonator of the previous point, but now the inductor has been replaced by a Josephson junction, effectively behaving as a non-linear inductor.

   (a) Find the Lagrangian of the CPB.
   (b) Find the associated quantized Hamiltonian.

Exercise 3 : Lossless transmission line

We now look into a very interesting example: a lossless transmission line (TL). A TL allows RF signals to be transmitted without significant amount of losses thanks to its property of confining electromagnetic fields between a central conductor and grounded outer shell (this is the case of the well known coaxial cable, for example). A TL can be modeled with an infinite series of fundamental cells constituted by an inductor within the inner conductor and a capacitor from the inner conductor to ground (see Fig 3).

The inductors model the inertia against changes in the electric current while the capacitors account for the electrostatic energy stored in the waveguide. The circuit is a discretized version of the guide where each capacitor and inductor accounts for a small segment Δx that is much smaller than the guided wavelengths. The properties of these elements depend on the capacitance and inductance per unit length:

$$C_i = c_i \Delta x, \qquad L_i = l_i \Delta x. \tag{2}$$

The goal is again to find the Lagrangian and quantized Hamiltonian for such an equivalent circuit describing a TL.

1. Find the Lagrangian of the system. Consider the branch fluxes along the inductors leftward oriented, meaning $\phi_{i+1\to i} = \phi_{i+1} - \phi_i$.

2. With the Legendre transformation, find the associated quantized Hamiltonian. Now, express it also in terms of c_i and l_i.

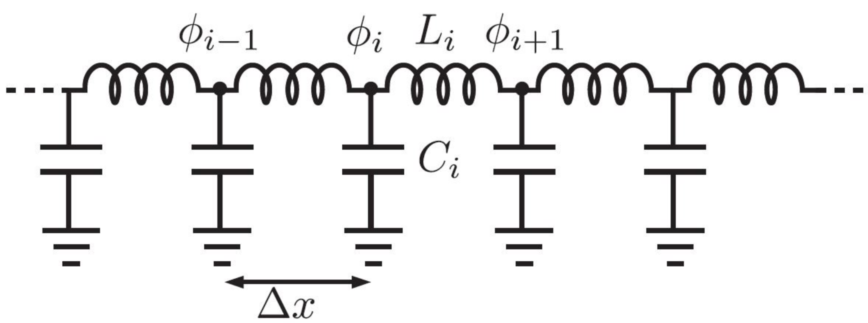

Figure 3: Circuit for a lossless transmission line.