Solid state systems for quantum information, Session 13

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Exercise 1 : Charge Stability Diagram of a Single Quantum Dot

During the lecture, you have seen how in a quantum dot electron transport is allowed only in very specific conditions. This is due to the fact the dot is so small that Coulomb repulsion cannot be neglected: adding an extra electron inside the dot costs energy, because Coulomb repulsion has to be overcome. The phenomenon for which transport is forbidden in certain regions is called *Coulomb blockade* (CB). To model the CB, there is no need of quantum mechanics. A simple capacitance model is sufficient, where the two tunneling barriers are modelled through "leaky" capacitors C_1 and C_2, i.e. dielectrics which allow the charge passage, whereas the action of the top gate, which controls the dot occupancy, through a conventional capacitor C_G, as pictured in Figure 1. $N|e|$ is the total number of electrons inside the island. This structure, where a central conductive island (i.e. the quantum dot) is separated from source and drain reservoirs by tunneling barriers, is called *Single Electron Transistor* (SET). The basic laws of electrostatic allow to write

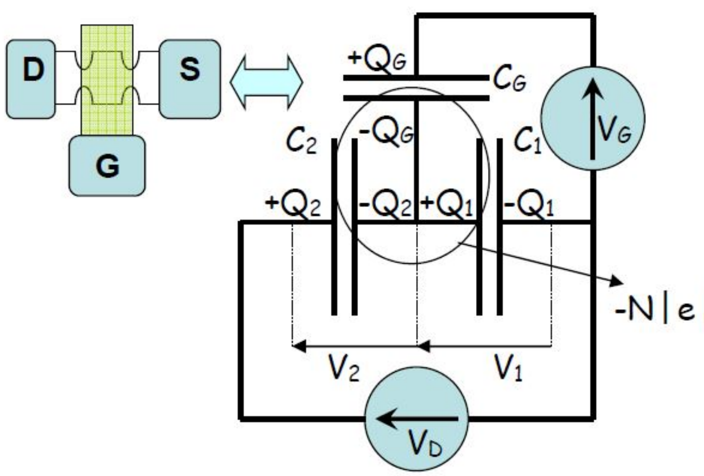


Figure 1: Capacitance model of the SET with a capacitor C_G between the gate and the conductive island and two capacitors C_1 and C_2 between the two source and drain reservoirs and the island. From *Electron transport in nanostructures and mesoscopic devices: an introduction, Ouisse Thierry*.

down all the equations to describe the circuit:

$$Q_G = C_G(V_G - V_1) \tag{1}$$
$$Q_1 = C_1 V_1 \tag{2}$$
$$Q_2 = C_2 V_2 \tag{3}$$
$$V_D = V_1 + V_2 \tag{4}$$
$$Q_1 - Q_2 - Q_G = -N|e|. \tag{5}$$

The electrostatic energy of the dot is given by:

$$E = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})d^3\vec{r} = \frac{1}{2}(Q_G V_G + Q_2 V_D - N|e|V_1 - Q_1 \cdot 0) = \frac{1}{2}(Q_G V_G - N|e|V_1 + Q_2 V_D). \quad (6)$$

Equations (1) to (5) allow to express the total electrostatic energy as a function of the different capacitances, V_G, V_D and N.

OPTIONAL: Find the the total electrostatic energy as a function of the different capacitances, V_G, V_D and N. The result should be the following:

$$E_{el} = \frac{C_1 C_G V_G^2 + C_1 C_2 V_D^2 + C_2 C_G (V_G - V_D)^2 + N^2 e^2}{2(C_1 + C_2 + C_G)}. \quad (7)$$

1. Which is the change in electrostatic energy, referred as *charging energy* or *addition energy* ΔE_C, when an extra electron is added from the source to the dot if already N electrons are inside?

   This last quantity is an important mean to understand when injecting electrons from source to the dot is energetically favorable. However, a complete energy balance should also take into account the energy spent by the two generators V_G and V_D:

   $$\Delta E_{gen} = \Delta E_G + \Delta E_D = -\delta Q_G V_G - \delta Q_2 V_D = C_G \delta V_1 V_G - C_2 \delta V_2 V_D. \quad (8)$$

   The "δ" means that only the N-dependent terms remain when subtracting the two terms V_1 and V_2 in the $N+1$ and N cases.

   OPTIONAL: Express this energy in terms of the different capacitances, V_G and V_D.

   You should get:

   $$\Delta E_{gen} = -\frac{|e|}{C_1 + C_2 + C_G}(C_G V_G + C_2 V_D). \quad (9)$$

2. Express the *total* change of charging energy when an extra electron is added from the source to the dot.

3. For which values of V_D is it energetically favorable to inject an electron from the source into the dot? Consider just the case where $V_D > 0$.

4. To measure a current through the dot (i.e. from source to drain), another condition must be fulfilled: it has to be energetically favorable also for an electron to tunnel from the dot into the drain. Which is the *total* change in electrostatic energy if the energy spent by the two generators is the following?

   $$\Delta E_{gen} = \Delta E_G + \Delta E_D = -\delta Q_G V_G + \delta Q_D V_D = \frac{|e|}{C_1 + C_2 + C_G}(C_G V_G - (C_1 + C_G)V_D). \quad (10)$$

5. For which values of V_D is it more favorable for an electron to tunnel from the dot into the drain? Consider again just the case where $V_D > 0$.

6. Draw the two expressions you just found in the previous points for V_D in a V_D vs V_G plane. What do you see? Focus on $N = -2, ..., 2$. Interpret the final result in terms of current through the dot.

7. So far, we have considered just the case where $V_D > 0$, because if $V_D < 0$ it is more energetically favorable to inject an electron from the drain into the dot and from the dot into the source, i.e. we get a negative current. If we repeat the same kind of computations of the previous points for this case, we get two new inequalities for V_D. By plotting also the equations of these two lines in the V_D vs V_G plane, we finally get the *"Coulomb diamonds"*. Can you guess what is the dot current dependence just on V_G, i.e. if you fix V_D by taking horizontal cuts in the V_G vs V_D plane?

8. Figure 2 shows what you should have got in the V_G vs V_D plot for three different values of N. This plot is called *charge stability diagram* of a single quantum dot, showing the typical *Coulomb diamonds*. Simply by looking at the diamonds, can you estimate how much is the charging energy of the dot? How much is the lever arm for the gate V_{gate} (remind: $\alpha = \frac{C_G}{C_{dot}}$). Which is the maximum temperature in the environment in order to observe "Coulomb Blockade" in this sample?
Hint: the expression of E_C found in point 1 is N-dependent. This is not so useful if we look at the diamonds and we do not know which value of N they represent. As a consequence, it diffused in contemporary literature to redefine the charging energy as following, to get something N-independent:

$$E_C = \Delta E_C(N+1) - \Delta E_C(N) = \frac{e^2}{C_{dot}}, \tag{11}$$

being $C_{dot} = C_G + C_1 + C_2$.

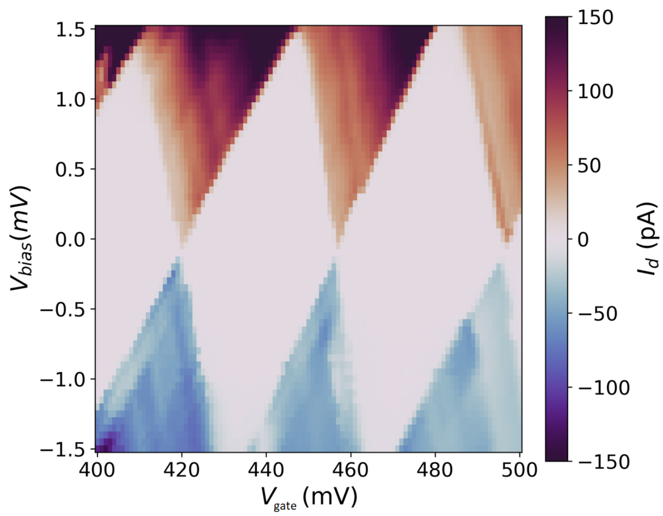


Figure 2: Charge stability diagram of a single Ge/SiGe quantum dot fabricated and measured at *Hybrid Quantum Circuits Laboratory*, EPFL. Spin Qubit 5 Conference, Pontresina 2022.

Exercise 2 : Electron in a Magnetic Field and Electron Spin Resonance (ESR)

In the previous exercise we have seen how it is possible to trap electrons inside a quantum dot in a semiconductor platform. We have also seen how the electron occupancy can be controlled by tuning the voltage of the top gate. Imagine to tune V_G in such a way we are inside the N=1 diamond. This means we have exactly one electron inside the dot. Now, if we apply a magnetic field, a splitting of the energy levels occurs according to the Zeeman effect.
The Hamiltonian of an electron inside a magnetic field can be written as follows:

$$H = g\mu_B \vec{B} \cdot \vec{S}, \tag{12}$$

where $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$ is the magnetic field and $\vec{S} = \frac{1}{2}\begin{pmatrix} \hat{\sigma_x} \\ \hat{\sigma_y} \\ \hat{\sigma_z} \end{pmatrix}$ the spin matrices.

1. A static magnetic field is applied in the z-direction to split the ground state energy of the electron in two sub-levels. Express explicitly the Hamiltonian in this case and derive the two eigenenergies. Which is the energy splitting between the two states? In order to exploit this system as a qubit, which is the constraint about this energy?
   For GaAs $g \approx -0.44$. Which is the minimum B_z that we have to apply? Suppose a temperature of 10 mK.

2. In order to manipulate the qubit state, we need to be able to do operations which flip the spin. In quantum mechanics, such as operations are off-diagonal elements of the Hamiltonian. In other words, we need an alternated magnetic field perpendicular to the static magnetic field. Let us suppose this magnetic field is in the x-direction. If the AC signal has a frequency ω and a phase ϕ, write the new Hamiltonian, also in the matrix representation by explicitly writing the Pauli matrices. Back to the GaAs platform, which frequency of the driving AC signal do we choose for a static $B_z = 1$ T if we want to drive at resonance?

3. Now we have off-diagonal terms which are time-dependent. Applying a unitary transformation allows us to move into a frame which rotates at the same frequency of the AC drive signal. Following the approach you have seen in the first exercise sheet, the Hamiltonian in the drive frame and after applying the rotating wave approximation to ignore fast rotating terms (RWA) should look like follows:

$$\hat{H} = -\frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & \Delta \end{pmatrix} \tag{13}$$

   Express Δ and Ω as a function of g and the magnetic field components B_x and B_z. Which is the value of B_x which allows to perform a π gate in 4 ns?

Exercise 3 : Identifying components on a real quantum dot device

In the two previous exercises you have seen how it is possible to confine single electrons in a gate-defined semiconducting quantum dot (QD) and how to manipulate the spin state by applying AC microwave signals. Fig. 3 shows how in practice a spin qubit device looks like on a GaAs/AlGaAs heterostructure.

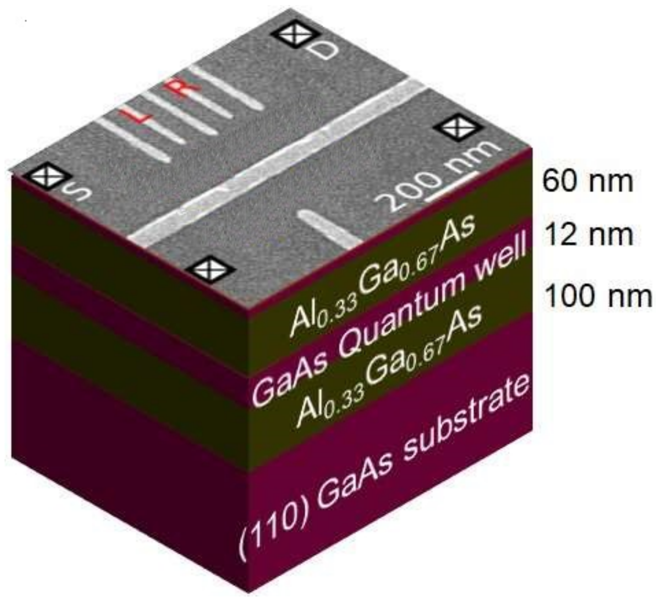


Figure 3: Top and cross view of a real quantum dot device on a GaAs/AlGaAs heterostructure.

1. Explain how is it possible to confine electrons in a 0D structure (a quantum dot). Start by trying to explain how a 2DEG can be formed in such a platform.
2. Identify all the components on top of the heterostructure. Which is the role of the single gate on the bottom-right part? How can you improve its sensitivity?
3. What is missing in this picture to perform spin manipulation?

Exercise 4 : Reflectometry technique for fast spin qubit readout

You have seen during the lecture that the readout of spin qubits relies on the spin-to-charge conversion through a charge sensor. However, the measurement bandwidth, in practice, is limited to 10-100 kHz by parasitic capacitances, RC filters used for the DC lines and the high resistance of the charge sensor ($\sim$ 100 kΩ). A way to overcome this issue and push the measurement bandwidth up to 1-10 MHz is to embed the charge sensor in a tank resonant circuit. An off-chip inductance L is usually soldered on the PCB and connected, for instance, to one of the ohmic contacts of the charge sensor. The parasitic capacitance of the bonding wire and the PCB C, together with the added inductance L, results in a resonating circuit (see Fig. 4). The idea of ohmic contact reflectometry is to send a RF tone through the tank circuit and analyse the reflected signal. When the charge sensor is in Coulomb blockade, its resistance is very large ($\gg G\Omega$) and the RF signal is almost completely reflected back because of the huge impedance mismatch. However, the value of L is chosen so that the impedance of the tank circuit at resonance matches the 50 Ω impedance of the transmission line when the charge sensor is parked at its most sensitive point. Here, the charge sensor resistance can vary from $R_{\text{SD}} \sim 10 - 500$ kΩ. When this condition is matched, the signal goes through the device and it is dissipated on the ohmic contact. As a consequence, the reflected signal S_{11} shows a pronounced dip at resonance. To readout the spin state, the charge sensor is parked at its most sensitive point (steepest point in current) and the resonator is probed at its resonance frequency (fixed frequency tone). When a tunneling event occurs in the spin qubits array, the chemical potential of the charge sensor changes and this kicks it out of its operating point. Its resistance, as a consequence, changes, the matching condition is not further fulfilled and the reflected signal changes both in magnitude and phase. By monitoring S_{11} when performing the readout, its change corresponds to a tunneling event, whereas if the signal does not change, no tunneling event happened.

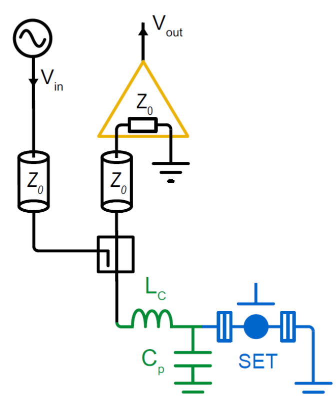


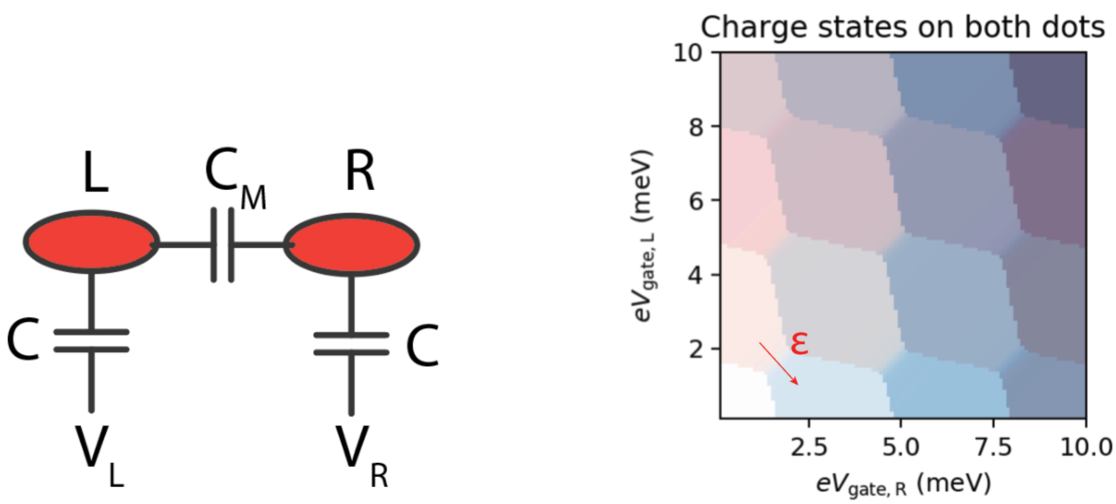
Figure 4: Schematic of ohmic reflectometry setup. The RF signal (V_{in}) travels down to the device, gets reflected and it is measured (V_{out}) after being amplified. The directional coupler makes sure the input signal goes only down to the device and that the reflected signal goes only to the amplifier.

1. Find the matching condition, i.e. $Z_{\text{tank}} = Z_0$, writing Z_{tank} as a function of L, C and R_{SD} *only*. In the end, simplify it by supposing $C^2 R_{\text{SD}}^2 \gg LC$.

2. Suppose $R_{\text{SD}} = 100$ kΩ, $C = 0.6$ pF and $Z_0 = 50$ Ω. Find the L which fulfills the matching condition and find the resonance frequency of the tank circuit.

Exercise 5 : Creating a charge qubit from two coupled dots

When two quantum dots, a "left" and a "right" dot, are coupled capacitively, an electron can hop from one dot to the other and vice versa. This phenomenon is the basis of semiconductor charge qubits, where the quantum information is encoded in the excess number of electrons of the two dots. For example, when having one more electron on the left dot (state $|L\rangle$) is energetically more favorable than having one more electron on the right dot (state $|R\rangle$), we can define the qubit states ($\{|0\rangle, |1\rangle\}$) such that $|0\rangle = |L\rangle$, and $|1\rangle = |R\rangle$. The simplest case is when there is only one electron in the double dot, i.e. $|L\rangle = (1, 0)$, and $|R\rangle = (0, 1)$, where $(N_L,\ N_R)$ denotes the state with N_L electron on the left dot and N_R electrons on the right dot.
The number of electrons on the dots can be tuned by the gate voltages of the dots, but, in contrast to uncoupled dots, the effect of the gates are not independent. This results in a charge stability diagram (the $(N_L,\ N_R)$ number of electrons on the two dots in the ground state of the system vs. gate voltages) which shows a honeycomb-like pattern instead of a chessboard pattern. In this problem, we will numerically explore this behaviour. Note that there is an example code on Moodle for two coupled dots, which could be useful for solving this problem.



(a) (b)

Figure 5: a) DQD simplified capacitance model. b) DQD charge stability diagram.

1. The figure above depicts two coupled dots, where each dot is coupled to a gate with capacitance C, the dots are coupled to each other with capacitance C_M, and the gate voltages are V_L and V_R. It can be shown [Rev. Mod. Phys. 75, 1 (2002)] that the total energy of the system is

$$H_{\text{DQD}} = E_C \cdot N_L^2 + E_C \cdot N_R^2 + 2E_{CM} N_L N_R - eV_L(\alpha N_L + \beta N_R) - eV_R(\beta N_L + \alpha N_R) \quad (14)$$

where the charging energies are $E_C = e^2 \frac{C}{2(C^2 - C_M^2)}$, and $E_{CM} = e^2 \frac{C_M}{2(C^2 - C_M^2)}$, while the coupling constants to the gates are $\alpha = \frac{C^2}{C^2 - C_M^2}$, and $\beta = \frac{CC_{MC}}{(C^2 - C_M^2)}$.
Plot the stability diagram of the system for $C = 50$ aF , $C_M = 10$ aF, with gate voltages corresponding to a few charge states (for example, $eV_{L/R} = 0...10$ meV).

2. So far, we described only the charging energy of the dot, i.e. the electron-electron interaction, but we have not taken into account the inter-dot tunneling effects. Thus, when electrons can jump between the two dots, we need to include an additional term in the Hamiltonian. This term can be expressed as

$$H_{\text{jump}} = -t(|N_L+1\rangle\langle N_L| \otimes |N_R-1\rangle\langle N_R| + |N_L-1\rangle\langle N_L| \otimes |N_R+1\rangle\langle N_R|) \tag{15}$$

because, for example, $|N_L+1\rangle\langle N_L| \otimes |N_R-1\rangle\langle N_R|$ describes the event when the number of electrons in the left dot is increased by one electron, while the number of electrons in the right dot is decreased by one electron, i.e., the electron jumped from the right to the left dot. Write down the H_{jump} operator in a matrix format in Python.
Hint: as $|N_L+1\rangle\langle N_L|$ is an off-diagonal matrix, it is a good idea to use sparse matrices. If you want to convert a sparse matrix M to a quitp Qobj object, use qt.Qobj(M).

3. Plot the stability diagram, when the tunneling amplitude is $t = 100$ μeV . If everything is correct, you will see that some of the boundaries between different states become smooth due to hybridization of the different charge states.

4. Plot the energy difference between the ground and the excited state, i.e. the qubit transition energy, along a line corresponding to the gate voltages that perpendicularly crosses the boundary of (1,0) and (0,1) states, for example, $eV_R = 1.5$ meV + ϵ, $eV_{GL} = 1.5$ meV - ϵ, where ϵ goes between -200 μeV and $+200$ μeV.