Solid state systems for quantum information, Session 10

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Exercise 1 : Two-qubit gates

We consider two transmon qubits, coupled via a capacitance C_0, see Fig. 1.

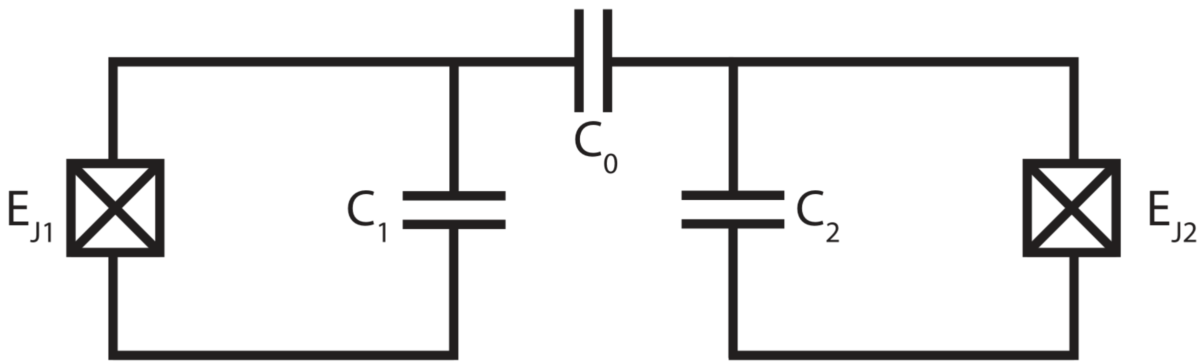

Figure 1: Electric circuit of two transmon qubits coupled via a capacitor C_0.

The effective two-qubit Hamiltonian describing the system reads

$$H/\hbar = \sum_{i=1}^{2} \left[\omega_i b_i^\dagger b_i + \frac{\alpha_i}{2} b_i^\dagger b_i^\dagger b_i b_i\right] + J(b_1^\dagger b_2 + b_2^\dagger b_1), \tag{1}$$

where ω_i is the frequency of the respective qubit, α_i is the anharmonicity, J is the coupling rate between two qubits and $b_i(b_i^\dagger)$ indicates the annihilation (creation) operator of the respective qubit.

1. Show, that the matrix representation of the Hamiltonian found in Eq. 1, with respect to the basis states $\{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle\}$, is:

$$H/\hbar = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_1 & J & 0 & 0 & 0 \\ 0 & J & \omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 + 2\omega_1 & \sqrt{2}J & 0 \\ 0 & 0 & 0 & \sqrt{2}J & \omega_1 + \omega_2 & \sqrt{2}J \\ 0 & 0 & 0 & 0 & \sqrt{2}J & \alpha_2 + 2\omega_2 \end{pmatrix} \tag{2}$$

   Use the fact that $b_1 |n, j\rangle = \sqrt{n}\,|(n-1), j\rangle$, and $b_2 |j, n\rangle = \sqrt{n}\,|j, (n-1)\rangle$, where $j \in \{0, 1, 2\}$.

2. Investigate how the system behaves if both qubits are on resonance, i.e. $\omega_1 = \omega_2 = 2\pi \cdot 6$ GHz. Use the prepared jupyter notebook to simulate the time evolution of the system for $J/2\pi = 8$ MHz, starting from $|\psi_0\rangle = |01\rangle$ and assume $\alpha_1 = \alpha_2 = -2\pi \cdot 300$ MHz. Plot the occupation probabilities of the states $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$ for $0 < t < \frac{3\pi}{2J}$. What is the two-qubit state you obtain at time $t = \frac{\pi}{4J}$?

3. Let us now consider the case $\omega_1 = \omega_2 - \alpha_1 = 2\pi \cdot 6$ GHz for $J/2\pi = 8$ MHz. Plot the evolution of the occupation probabilities, starting from an initial state $|11\rangle$ for $0 < t < \frac{3\pi}{2J}$. After ehat time t_1 has the system returned back to its initial state $|11\rangle$? Why does the system return faster to the initial state $|11\rangle$ than it returned to the initial state $|01\rangle$ in 2.

4. Plot the time evolution of the system with $J/2\pi = 8$ MHz where $\omega_1 = \omega_2 - \alpha_1 - \delta = 2\pi \cdot 6$ MHz, for detunings $\delta/2\pi \in [-50, 50]$ MHz. For all detunings, find the time $t_i(\delta)$ you need to evolve the system in order to return to the state $|11\rangle$ (when starting in $|11\rangle$). Plot this time $t_i(\delta)$ vs the detuning.

Exercise 2 : Virtual Photon Exchange

When two atoms are coupled to the same cavity, you have two Jaynes-Cummings systems. The full Hamiltonian is given by:

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{3}$$

with
$$\hat{H}_0 = \hbar\omega_c\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,1} + \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,2} \tag{4}$$

and
$$\hat{H}_I = \hbar g(\hat{a}^\dagger\hat{\sigma}_{-,1} + \hat{a}\hat{\sigma}_{+,1}) + \hbar g(\hat{a}^\dagger\hat{\sigma}_{-,2} + \hat{a}\hat{\sigma}_{+,2}), \tag{5}$$

where ω_c is the frequency of cavity photons, $\hat{a}^\dagger$ and $\hat{a}$ create and anihilate cavity photons, ω_a is the frequency spacing of the atomic levels, g quantifies the coupling strength, and $\hat{\sigma}_{\pm,i} = \frac{1}{2}(\hat{\sigma}_{x,i} \pm i\hat{\sigma}_{y,i})$ for atom i. Our goal is to rewrite this Hamiltonian as the following (notice that, in the end, we do not consider the cavity anymore):

$$\hat{H} \approx \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,1} + \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,2} + J(\hat{\sigma}_{-,1}\hat{\sigma}_{+,2} + \hat{\sigma}_{+,1}\hat{\sigma}_{-,2}) \tag{6}$$

1. We will in the following assume that $g \ll |\Delta| = |\omega_a - \omega_c|$. Why do you expect that this assumption will allow us to eventually not consider the cavity in the Hamiltonian?

2. Show that the commutator

   $$[(\hat{a}\hat{\sigma}_{-,i} - \hat{a}^\dagger\hat{\sigma}_{-,i}), \hat{H}] = -\hbar\Delta(\hat{a}^\dagger\hat{\sigma}_{-,i} + \hat{a}\hat{\sigma}_{+,i}) + \hbar g(2\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \hat{\sigma}_{z,i}) + \hbar g(\hat{\sigma}_{-,i}\hat{\sigma}_{+,j} + \hat{\sigma}_{+,i}\hat{\sigma}_{-,j}) \tag{7}$$

   for any i (with $j \neq i$).

3. The idea is now to perform a unitary transformation $H \to UHU^\dagger$ with the unitary $U = \exp\left(\sum_i \frac{g}{\Delta}(\hat{a}\hat{\sigma}_{+,i} - \hat{a}^\dagger\hat{\sigma}_{-,i})\right)$. It turns out, that this unitary transformation approximately diagonalizes the initial Hamiltonian.

   You can now use the "Baker-Campbell-Hausdorf" formula:

   $$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots + \frac{1}{n!}\overbrace{[A,[A,\ldots[A}^{nA's},B]\ldots] + \ldots \tag{8}$$

   Calculate the transformed Hamiltonian up to second order in (g/Δ)

   Hint: You will need some terms from the $\frac{1}{2}[A,[A,B]]$-terms.

4. The resulting Hamiltonian now contains both cavity and qubit terms. Bring the transformed Hamiltonian to the form in Eq. 6 by assuming that the cavity is in the ground state at all times. What is J? Interpret the fact that the qubit interact while we explicitly assumed the cavity to be in the ground state.

5. Since the qubits are on resonance, it is enough to just consider the Hamiltonian $H = J(\hat{\sigma}_{-,1}\hat{\sigma}_{+,2} + \hat{\sigma}_{+,1}\hat{\sigma}_{-,2})$. If the qubits start in state $|1\rangle|0\rangle$, what is the state after $t = \pi/J$ and after $t = \pi/(2J)$?

Exercise 3 : Superconducting coupling bus

This problem is about the paper "Coupling superconducting qubits via a cavity bus" by J. Majer et al (2007). You can find the paper on Moodle.
In the previous problem, you derived the main formula of this paper. Here, we will discuss some experimental aspects of this work. This paper was a milestone in the field of superconducting qubits as it was the first to demonstrate that superconducting qubits can interact through coupling bus.

1. In this paper, what is the qubit-cavity coupling strength for the two qubits ?
2. If we consider one of the qubits detuned from the other, we can write the Hamiltonian

   $$\hat{H} \approx \frac{\hbar\delta}{2}\hat{\sigma}_{z,1} + J(\hat{\sigma}_{-,1}\hat{\sigma}_{+,2} + \hat{\sigma}_{+,1}\hat{\sigma}_{-,2}) \tag{9}$$

   Find the eigenstates and eigenenergies for this Hamiltonian, first for $\delta = 0$ and then for any δ?

   Hint: You will have $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$ as trivial eigenstates and then two "dressed" eigenstates as linear combinations of $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$.
3. What is the minimum energy difference between the two dressed eigenstates? How can you see this value in Figure 2?
4. Explain, conceptually, all additional features you see in Figure 2.
5. The paper mentions the a.c. Stark terms with the strength χ_i. Explain the effect of these terms when adding photons to the cavity.
6. Explain how the a.c. Stark effect was used to change the effective Hamiltonian in Figure 4.
7. Explain the oscillations in Figure 4.
8. The oscillation frequency follow a parabola in Figure 4 (d). Use the results of problem 3. to explain the parabolic shape.
9. Bonus question: While the coupling through a bus resonators, as in this paper, is widely used today in superconducting qubit labs at ETH, Google, Delft and many more, it is now possible to tune the frequency of qubits directly instead of using the a.c. Stark shift. What is the downsides of the a.c. Stark shift methods as demonstrated in this paper?