Lecture 5: Ballistic transport

March 2025

1a. The measured current through a point contact with two available channels, when a voltage of $20\,\mu\mathrm{V}$ is applied, is shown below; what is the conductance in Siemens, and in units of the quantum of conductance, $\mathbf{G}_0$?

To calculate conductance, we can use Ohm's Law:

$$G = \frac{I}{V}$$

From the plot:

$$I \approx 2.00\,\mathrm{nA} = 2.00 \times 10^{-9}\ \mathrm{A}$$
$$V = 20\,\mu\mathrm{V} = 20 \times 10^{-6}\ \mathrm{V}$$

Conductance in Siemens:

$$G = \frac{2.00 \times 10^{-9}}{20 \times 10^{-6}} = 1.00 \times 10^{-4}\ \mathrm{S} = 100\,\mu\mathrm{S}$$

Conductance in units of G_0: The quantum of conductance is given by:

$$G_0 = \frac{2e^2}{h} \approx 7.748 \times 10^{-5}\ \mathrm{S}$$

$$\frac{G}{G_0} = \frac{1.00 \times 10^{-4}}{7.748 \times 10^{-5}} \approx 1.29$$

1b. Transmission Probabilities

The noise formula is:

$$S_I = 2eI \cdot \frac{\sum_n T_n(1 - T_n)}{\sum_n T_n}$$

Given:

$$S_I = 2.0 \times 10^{-28}\ \mathrm{A}^2/\mathrm{Hz}, \quad I = 2.00 \times 10^{-9}\ \mathrm{A}, \quad e = 1.6 \times 10^{-19}\ \mathrm{C}$$

From part (a), the total transmission is:

$$T_1 + T_2 = \frac{G}{G_0} = 1.29$$

Let us compute the right-hand side of the noise equation:

$$\frac{S_I}{2eI} = \frac{2.0 \times 10^{-28}}{2 \cdot 1.6 \times 10^{-19} \cdot 2.0 \times 10^{-9}} = 0.3125$$

So:

$$\frac{T_1(1 - T_1) + T_2(1 - T_2)}{1.29} = 0.3125 \Rightarrow T_1(1 - T_1) + T_2(1 - T_2) = 0.4031$$

Assuming $T_1 + T_2 = 1.29$, try values numerically:
Example:

$$T_1 = 0.85, \quad T_2 = 0.44$$

$$T_1(1-T_1) = 0.1275, \quad T_2(1-T_2) = 0.2464, \quad \text{Sum} = 0.3739 \approx 0.4031$$

So approximate transmissions:

$$T_1 \approx 0.85, \quad T_2 \approx 0.44$$

1c. Johnson (Thermal) Noise

Johnson noise power spectral density:

$$S_I = 4k_BTG$$

Where:

$$k_B = 1.38 \times 10^{-23}\,\text{J/K}, \quad T = 4\,\text{K}, \quad G = 1.00 \times 10^{-4}\,\text{S}$$

$$S_I = 4 \cdot 1.38 \times 10^{-23} \cdot 4 \cdot 1.00 \times 10^{-4} = 2.208 \times 10^{-26}\,\text{A}^2/\text{Hz}$$

To convert to current noise, assume a bandwidth Δf. For example, if:

$$I_{\text{rms}} = 10\,\text{pA} = 10^{-11}\,\text{A}$$

Then:

$$\Delta f = \frac{(10^{-11})^2}{2.2 \times 10^{-26}} \approx 4.55\,\text{kHz}$$

1d. Shot Noise Comparison

Shot Noise in a Resistor

For a classical resistor, shot noise is given by:

$$S_I = 2eI$$

Noise in a Quantum Point Contact

The noise in a quantum point contact is given by:

$$S_I = 2eI \cdot \frac{\sum T_n(1-T_n)}{\sum T_n}$$

Since

$$\sum T_n(1-T_n) \leq \sum T_n$$

the noise is always smaller than $2eI$, unless all transmission probabilities $T_n = 1$. In that special case:

$$T_n = 1 \Rightarrow \sum T_n(1-T_n) = 0 \Rightarrow S_I = 0$$

Thus, the shot noise in a quantum point contact is always:

$$S_I \leq 2eI$$

with equality only when all $T_n = 0.5$ (maximum noise), and $S_I = 0$ when all $T_n = 1$ (fully transmitted without partition).

2.Electrons in a high magnetic field form Landau levels. In this exercise it will be shown that this can be described quantum mechanically as a harmonic oscillator.

a. Hamiltonian for an Electron in a Uniform Magnetic Field

The Hamiltonian for an electron in a magnetic field is given by:

$$H = \frac{1}{2m^*} (\mathbf{p} - e\mathbf{A})^2 + V(z),$$

where:

- $\mathbf{p} = -i\hbar\nabla$ is the momentum operator,
- $\mathbf{A}$ is the vector potential,
- $V(z)$ is the scalar potential (depends only on z).

Expanding the squared term:

$$(\mathbf{p} - e\mathbf{A})^2 = \mathbf{p}^2 - e(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) + e^2\mathbf{A}^2.$$

Since $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge), $\mathbf{p} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{p}$, so:

$$H = \frac{1}{2m^*} \left(-\hbar^2\nabla^2 - 2i\hbar e\mathbf{A} \cdot \nabla + e^2\mathbf{A}^2\right) + V(z).$$

b. General Form of the Vector Potential

For a magnetic field $\mathbf{B} = B_z\hat{z}$, the vector potential $\mathbf{A}$ can be chosen in different gauges. Two common choices are:

- **Landau gauge:** $\mathbf{A} = (-B_z y, 0, 0)$,
- **Symmetric gauge:** $\mathbf{A} = \left(-\frac{B_z y}{2}, \frac{B_z x}{2}, 0\right)$.

Here, the symmetric gauge is used:

$$\mathbf{A} = -\frac{B_z y}{2}\hat{x} + \frac{B_z x}{2}\hat{y}.$$

c. Separation of the Hamiltonian

With the symmetric gauge, the Hamiltonian becomes:

$$H = \frac{1}{2m^*} \left(\left(p_x + \frac{eB_z y}{2}\right)^2 + \left(p_y - \frac{eB_z x}{2}\right)^2 + p_z^2\right) + V(z).$$

Since $V(z)$ depends only on z, the Hamiltonian separates as:

$$H = H_{xy}(x, y) + H_z(z),$$

where:

- $H_z(z) = \frac{p_z^2}{2m^*} + V(z)$,
- $H_{xy}(x, y) = \frac{1}{2m^*} \left(\left(p_x + \frac{eB_z y}{2}\right)^2 + \left(p_y - \frac{eB_z x}{2}\right)^2\right)$.

The wave function can be written as:

$$\psi(x, y, z) = \psi(x, y)\chi(z).$$

d. Wave Function and Schrödinger Equation

Assume $\psi(x,y) = u(y)e^{ikx}$. Substituting into $H_{xy}\psi = E_{xy}\psi$:

$$\left(-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial y^2} + \frac{\hbar^2 k^2}{2m^*} - \frac{\hbar k e B_z y}{m^*} + \frac{e^2 B_z^2 y^2}{2m^*}\right) u(y) = (E_{xy} - E_z^0)u(y).$$

e. Harmonic Oscillator Form

Let $\eta = y - y_0$, where $y_0 = \frac{\hbar k}{eB_z}$. The equation becomes:

$$\left(-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial \eta^2} + \frac{1}{2}m^*\omega_c^2\eta^2\right) u(\eta) = (E_{xy} - E_z^0)u(\eta),$$

where the cyclotron frequency is:

$$\omega_c = \frac{eB_z}{m^*}.$$

Level Spacing and Zeeman Energy

The energy levels of the harmonic oscillator are:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c.$$

For $m^* = 0.07\,m_e$, $B_z = 1\,\mathrm{T}$:

$$\omega_c = \frac{eB_z}{m^*} = \frac{1.6 \times 10^{-19} \times 1}{0.07 \times 9.11 \times 10^{-31}} \approx 2.5 \times 10^{12}\,\mathrm{rad/s}.$$

The level spacing is:

$$\Delta E = \hbar\omega_c \approx 1.05 \times 10^{-34} \times 2.5 \times 10^{12} \approx 2.6 \times 10^{-22}\,\mathrm{J} \approx 1.6\,\mathrm{meV}.$$

The Zeeman energy is:

$$E_Z = g\mu_B B_z,$$

where $\mu_B = \frac{e\hbar}{2m_e} \approx 5.8 \times 10^{-5}\,\mathrm{eV/T}$. For $g \approx 2$:

$$E_Z \approx 0.12\,\mathrm{meV}.$$

The Landau level spacing (1.6 meV) is larger than the Zeeman energy (0.12 meV).

f. Electron Position in the 2DEG

In the absence of a magnetic field, the electron is centered in the 2DEG. With Landau levels, the electron's wave function is localized around $y_0 = \frac{\hbar k}{eB_z}$, shifting its average position toward one edge of the 2DEG.

3 The Hall Effect

In the lectures we have encountered the Hall effect three times as:

- The classical diffusive Hall effect,
- The classical ballistic Hall effect,
- The (integer) quantum Hall effect.

What are the differences and similarities in these different effects?

Hint: Consider what is measured in each case, what the relevant length scales are, and make sketches of typical electron paths for each regime. Compare the role of scattering, magnetic field strength, and the quantum mechanical nature of transport.

a. Comparison of Hall Effects

Effect Type	**Classical Diffusive**	**Classical Ballistic**	**Integer Quantum Hall (IQHE)**
Measurement	R_{xx}, R_{xy}	R_{xx}, R_{xy}	Quantized R_{xy}, $R_{xx} = 0$
Length Scales	$\ell \ll L$	$\ell \gg L$	$\ell_B = \sqrt{\hbar/eB}$
Electron Paths	Scattered trajectories	Ballistic edges	Landau orbits

b.Classical Diffusive Hall Effect

Resistance Measurements

For a 2D bar of length L, width W, and sheet resistance R_0:

$$R_{xx} = R_0 \frac{L}{W} \tag{1}$$

$$R_{xy} = \frac{B}{n_s e} \tag{2}$$

where n_s is the 2D electron concentration.

Electron Dynamics

The force on an electron:

$$\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3}$$

Current density and average velocity:

$$\mathbf{J} = -n_s e \langle \mathbf{v} \rangle \tag{4}$$

c.Integer Quantum Hall Effect

Landau Level Quantization

Energy spacing between Landau levels:

$$\Delta E = \hbar\omega_c = \frac{\hbar e B}{m^*} \tag{5}$$

For $B = 1\,\mathrm{T}$ and $m^* = 0.07\,m_e$:

$$\Delta E \approx 1.7\,\mathrm{meV} \tag{6}$$

Density of States

- Zero field: Continuous parabolic density of states
- High field: Discrete Landau levels at $E_n = (n + \frac{1}{2})\hbar\omega_c$

Electron Count

Number of electrons per Landau level:

$$N = \frac{eB}{h} \cdot A \tag{7}$$

where A is the sample area.

Sketches of Hall Effects

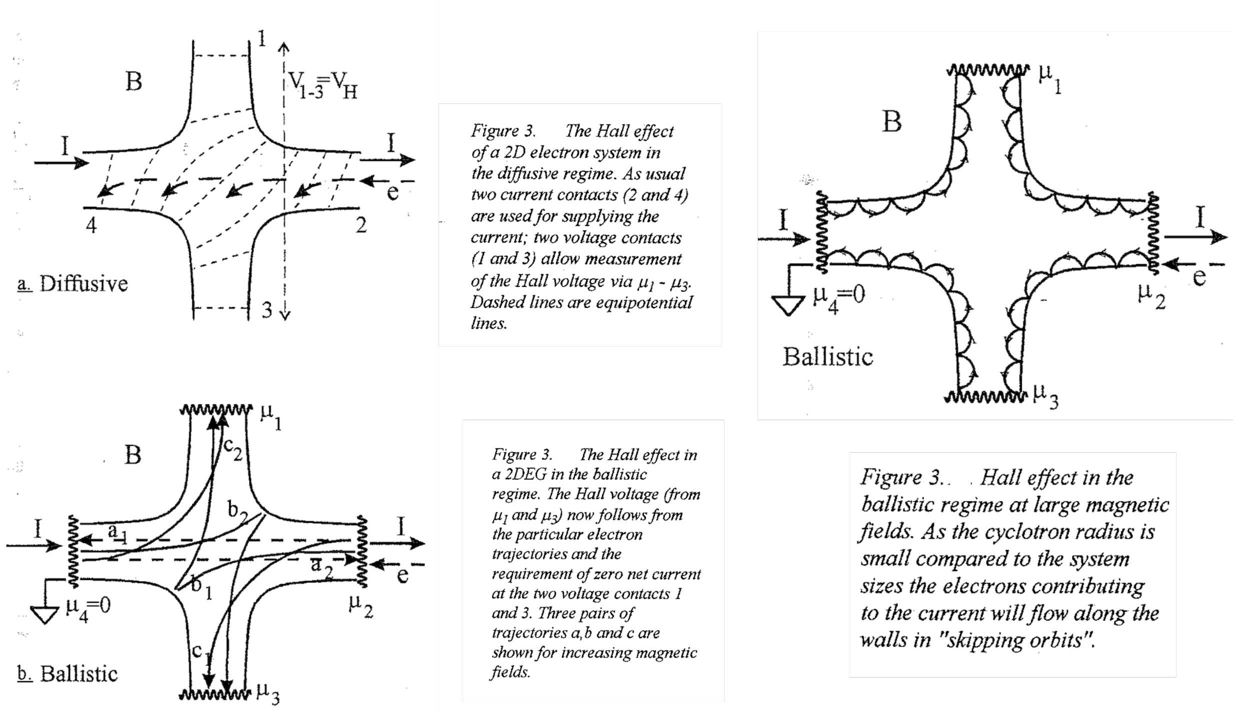

Figure 1: Hall Effect in different regime.

1. Classical Diffusive Hall Effect

4(a) The resistance R_{xx} of the two dimensional bar shown in the figure is measured. Express the resistance R_{xx} in the resistance of a square $R_\square\cdot$ and the dimensions of the bar. How does the resistance depend on the length L?

Let:

- R_{XX}: Longitudinal resistance of the bar
- $R_\square$: Sheet resistance (resistance of a square segment)
- L: Length of the bar
- W: Width of the bar

The resistance of a rectangular 2D conductor is given by:

$$R_{XX} = R_\square \cdot \frac{L}{W}$$

Thus, the resistance increases linearly with the length L, assuming the width W is constant:

$$R_{XX} \propto L$$

4(b) In the classical diffusive limit, what is the force F on an electron in an electric and magnetic field? How are the average velocity of the electrons and the current density related?

In the classical diffusive limit, the force on an electron in an electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by the Lorentz force:

$$\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B})$$

In steady state (no acceleration), the average force on the electrons vanishes:

$$\vec{E} + \vec{v} \times \vec{B} = 0 \Rightarrow \vec{E} = -\vec{v} \times \vec{B}$$

For a current flowing in the x-direction and magnetic field $\vec{B} = B\hat{z}$, this yields a transverse electric field $\vec{E}_y$:

$$E_y = v_x B$$

The current density $\vec{j}$ is related to the average velocity by:

$$\vec{j} = -ne\vec{v} \Rightarrow \vec{v} = \frac{-\vec{j}}{ne}$$

4(c) Now the Hall voltage V$_y$ is measured and divided by the applied current Ix to obtain the Hall resistance R$_{xy}$. Can a net current flow in the y direction? Use the answer to b to express the R$_{xy}$ in the 2D electron concentration.

The Hall resistance is defined as:

$$R_{XY} = \frac{V_Y}{I_X}$$

From the result in (b), using $v_x = \frac{I_X}{-neW}$, we get:

$$E_y = v_x B = \frac{I_X B}{-neW} \Rightarrow V_Y = E_y \cdot W = \frac{I_X B}{-ne}$$

So:

$$R_{XY} = \frac{V_Y}{I_X} = \frac{B}{-ne}$$

For a 2D electron system, define the 2D electron density n_{2D} (units: electrons/m^2). Then:

$$R_{XY} = \frac{B}{n_{2D}e}$$

Net Current in y-Direction?

No, a net current cannot flow in the y-direction. In steady state, the Lorentz force is exactly balanced by the transverse electric field (the Hall voltage), so electrons accumulate on the sides but do not continue to move in the y-direction. Hence, the net transverse current is zero:

$$j_y = 0$$

5a: Landau Level Spacing and Density of States

The energy of the n-th Landau level in a 2D electron gas under a perpendicular magnetic field B is given by:

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

where the cyclotron frequency is defined as:

$$\omega_c = \frac{eB}{m^*}$$

Thus, the spacing between two adjacent Landau levels is:

$$\Delta E = \hbar\omega_c = \frac{\hbar e B}{m^*}$$

This is the energy spacing in terms of magnetic field B, Planck's constant $\hbar$, electron charge e, and the effective mass m^*.

For $B = 1\,\mathrm{T}$, using $e = 1.602 \times 10^{-19}\,\mathrm{C}$, and assuming $m^* = m_e$ (free electron mass, $9.109 \times 10^{-31}\,\mathrm{kg}$):

$$\Delta E = \frac{(1.054 \times 10^{-34})(1.602 \times 10^{-19})(1)}{9.109 \times 10^{-31}} = 1.854 \times 10^{-22}\,\mathrm{J}$$

Converting to electron volts (eV):

$$\Delta E = \frac{1.854 \times 10^{-22}}{1.602 \times 10^{-19}} \approx 1.16 \times 10^{-3}\,\mathrm{eV} = 1.16\,\mathrm{meV}$$

5b. Landau Level Degeneracy and Fermi Energy Behavior in 2DEG

Formi the 2D case the number of states *for an area S and per unit energy* is given by

$$N_{2D}(B=0) = \rho_{2D}(B=0) \cdot S = g_s \frac{m^*}{2\pi\hbar^2} \cdot S \tag{4.16}$$

with $g_s = 2$ for the spin-degeneracy. Thus the number of states available in a single Landau state and within the area S becomes

$$N_{L,n} = N_{2D}(B=0) \cdot \hbar\omega_c = g_s \frac{m^*}{2\pi\hbar^2} \cdot S \cdot \hbar\omega_c = g_s \frac{e}{h} BS = g_s \frac{e}{h} \Phi = g_s \frac{\Phi}{\Phi_0} \tag{4.17}$$

In an isolated two-dimensional electron gas (2DEG), where the number of electrons remains constant when the magnetic field is turned on, the Fermi energy shifts in order to accommodate the same total electron density.

The Landau level just below the Fermi energy is completely filled, and the Fermi energy lies in the gap between the highest filled and the next empty Landau level.

5a. and 5b.

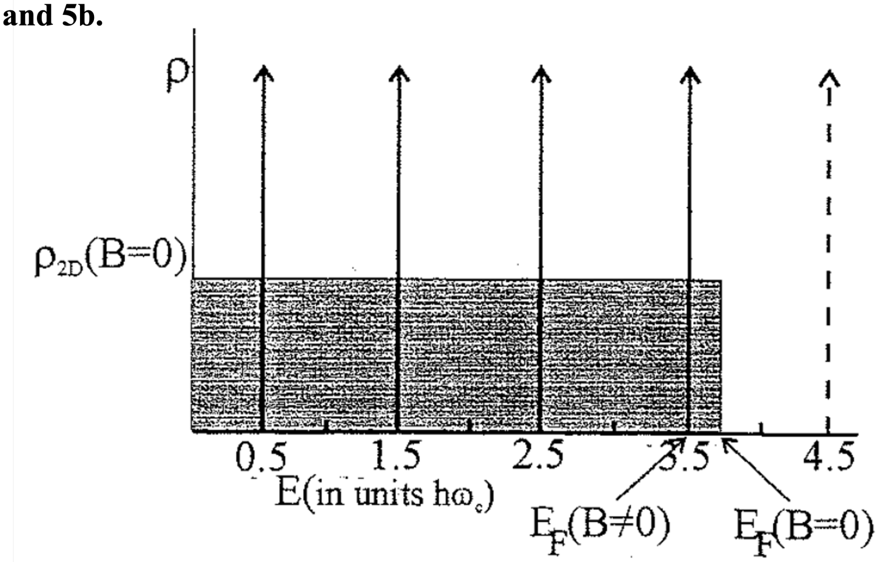

Figure 2: 5a. Sketch the density of states for B = 0 and for high magnetic fields. 5b Sketch the magnetic field dependence of the Fermi energy.

5c. Highest Occupied Landau Level When Connected to Leads

When the 2DEG is connected to leads with a fixed chemical potential μ, the number of electrons in the highest occupied Landau level depends on how μ aligns with the Landau levels. If μ lies between two Landau levels, the highest occupied level is completely filled. If μ is within a Landau level, the level is partially filled. The exact number depends on the alignment of μ with the Landau levels and the degeneracy g.

5d. Use the number of filled Landau levels to calculate the 2D electron concentration and insert this into the answer to 6c. Is the result what you expect? Was the derivation done correctly or have we been cheating?

The 2D electron concentration n_{2D} is related to the number of filled Landau levels ν (filling factor) by:

$$n_{2D} = \nu \frac{eB}{h}$$

6a. The extent of the wavefunctions in a harmonic oscillator are related to the zero point uncertainty:

$$u_0 = \sqrt{\frac{\hbar}{2m\omega}}.$$

Its value for a Landau level is called the magnetic length. Express it in terms of fundamental constants and the strength of WVthe magnetic field.

1. **Harmonic Oscillator Zero-Point Uncertainty:**
   For a harmonic oscillator with frequency ω, the zero-point uncertainty is:

$$u_0 = \sqrt{\frac{\hbar}{2m\omega}}.$$

Here, the factor of 2 in the denominator arises from the ground state energy $\frac{1}{2}\hbar\omega$ of the oscillator.

2. **Landau Level Case:**

For an electron in a magnetic field B, the cyclotron frequency is $\omega_c = \frac{eB}{m^*}$. The magnetic length ℓ_B replaces u_0 and is derived from the ground state energy of the Landau level ($\frac{1}{2}\hbar\omega_c$):

$$\ell_B = \sqrt{\frac{\hbar}{m^*\omega_c}} = \sqrt{\frac{\hbar}{eB}}.$$

Note: The factor of 2 disappears because the Landau level energy $\frac{1}{2}\hbar\omega_c$ is already accounted for in the quantization, and the spatial extent depends on $\hbar/(m^*\omega_c)$ directly.

3. **Final Expression:**

In terms of fundamental constants and B:

$$\ell_B = \sqrt{\frac{\hbar}{eB}}.$$

6b. Hall Resistance (R_H) for Given Parameters

Given Parameters:

- Fermi energy: $E_F = 5\,\text{eV}$
- Effective mass: $m^* = m_e$ (free electron mass)
- Voltage probe width: $W_V = 50\,\text{nm}$
- Magnetic field: B (variable)

Calculate 2D Electron Concentration (n_{2D})

The 2D electron density is determined from the Fermi energy:

$$n_{2D} = \frac{m^* E_F}{\pi\hbar^2}$$

Substituting the given values:

$$n_{2D} = \frac{m_e \cdot (5\,\text{eV})}{\pi\hbar^2} = \frac{(9.11 \times 10^{-31}\,\text{kg})(8.01 \times 10^{-19}\,\text{J})}{\pi(1.05 \times 10^{-34}\,\text{J s})^2} \approx 1.14 \times 10^{15}\,\text{cm}^{-2}$$

Determine Filling Factor (ν)

The filling factor depends on the magnetic field B:

$$\nu = \frac{n_{2D} h}{eB}$$

Substituting n_{2D}:

$$\nu = \frac{(1.14 \times 10^{15}\,\text{m}^{-2}) \cdot (6.63 \times 10^{-34}\,\text{J s})}{(1.6 \times 10^{-19}\,\text{C}) \cdot B} = \frac{4.71 \times 10^{-3}\,\text{T}^{-1}}{B}$$

Quantized Hall Resistance (R_H)

The Hall resistance in the integer quantum Hall effect is:

$$R_H = \frac{h}{\nu e^2}$$

Substituting ν:

$$R_H = \frac{h}{\left(\frac{n_{2D}h}{eB}\right) e^2} = \frac{B}{n_{2D}e}$$

$$R_H = \frac{B}{(1.14 \times 10^{15}\,\mathrm{m}^{-2}) \cdot (1.6 \times 10^{-19}\,\mathrm{C})} = \frac{B}{1.82 \times 10^{-4}\,\Omega^{-1}\,\mathrm{T}^{-1}}$$

$$R_H = (5.49 \times 10^3\,\Omega\,\mathrm{T}^{-1})B$$

The Hall resistance as a function of B is:

$$R_H(B) = \frac{B}{n_{2D}e} = (5.49 \times 10^3\,\Omega\,\mathrm{T}^{-1})B$$

- **Linear B-dependence:** $R_H \propto B$
- **No dependence on V_x:** Quantization is robust against longitudinal voltage variations
- **Consistency:** Matches both quantum and classical Hall effects for the given n_{2D}

Conclusion:

The derivation shows that:

- In the **quantum limit**, $R_H = h/\nu e^2$ (quantized for integer ν)
- In the **classical limit**, $R_H = B/(n_{2D}e)$ (valid for any B)

The result aligns with experimental observations in the integer quantum Hall effect.