Lecture 3: Ballistic transport

March 2025

1a. What is the order of the length-scales for classical ballistic transport: L, λ_F, le and lϕ??

The classical ballistic regime is characterised by the following relation between length scales:

$$\lambda_F \ll L < l_i, l_e$$

i.e., the size of the system, L, is taken to be considerably smaller than any of the characteristic scattering lengths. The Fermi wavelength is assumed to play no role at all.

1b. How does the resistance of a channel change when the length is increased for a quasi and true ballistic channel?

The resistance of a channel behaves differently when the length is increased, depending on whether the transport is quasi-ballistic or truly ballistic. Below is a detailed explanation for both cases:

1. True Ballistic Transport

In a truly ballistic channel, electrons travel through the conductor without any scattering events. This means that the mean free path (l_e) is much larger than the length of the channel (L).

- **Resistance in True Ballistic Transport:**
  - The resistance is primarily determined by the contacts and the quantum conductance of the channel.
  - The resistance does not depend on the length of the channel because there are no scattering events within the channel.
  - The resistance R can be expressed as:

$$R = \frac{h}{2e^2} \cdot \frac{1}{N}$$

    where h is Planck's constant, e is the elementary charge, and N is the number of conducting channels (modes).

- **Effect of Increasing Length:**
  - In a truly ballistic channel, increasing the length L does not change the resistance because the electrons do not experience any scattering within the channel.

2. Quasi-Ballistic Transport

In a quasi-ballistic channel, electrons experience some scattering events, but the mean free path (l_e) is still relatively large compared to the length of the channel (L).

- **Resistance in Quasi-Ballistic Transport:**
  - The resistance has contributions from both the contacts and the scattering within the channel.
  - The resistance R can be expressed as:

$$R = \frac{h}{2e^2} \cdot \frac{1}{N} + R_{\text{scattering}}$$

    where $R_{\text{scattering}}$ is the additional resistance due to scattering events within the channel.

- **Effect of Increasing Length:**
  - As the length L of the channel increases, the probability of scattering events increases, leading to an increase in $R_{\text{scattering}}$.
  - Therefore, the resistance of a quasi-ballistic channel increases with increasing length, but the increase is not as pronounced as in a diffusive transport regime where scattering is frequent.

1c. A ballistic nanotube is measured in a four-terminal geometry. What is larger, the two-terminal or the four-terminal resistance and what causes the difference.

- **Two-terminal resistance:** This includes both the intrinsic resistance of the nanotube and the contact resistance between the nanotube and the electrodes. The two-terminal resistance is typically larger because it accounts for the resistance at the contacts, which can be significant in nanoscale devices.

- **Four-terminal resistance:** This measurement is designed to eliminate the contact resistance by using separate pairs of electrodes for current injection and voltage measurement. The four-terminal resistance is therefore smaller than the two-terminal resistance because it only measures the intrinsic resistance of the nanotube.

1d. First the two-terminal resistance is measured. What values for the four point-measurements can you expect?

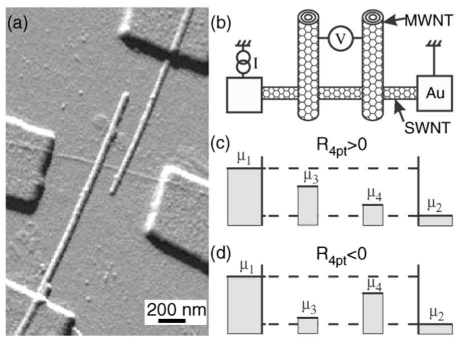

Figure 1: (a) Atomic force microscopy image of a SWNT con- tacted by 2 MWNTs and 2 Au electrodes (b) Schematic of the R4pt measurement. (c), (d) Levels of the electrochemical potential for the 4 electrodes that give a positive R4pt in (c) and a negative R4pt in (d).

The current I_α in each electrode is related to the electrochemical potential μ_β of other electrodes by

$$I_\alpha = \frac{4e^2}{h} \sum_\beta T_{\beta\alpha}\mu_\alpha - T_{\alpha\beta}\mu_\beta$$

with $T_{\alpha\beta}$ the total transmission between the α and the β electrodes [Figs. 1(c) and 1(d)]. The condition $I_3 = 0$ for a voltage probe gives

$$\mu_3 = \frac{T_{31}\mu_1 + T_{32}\mu_2}{T_{31} + T_{32}}.$$

The transmission between electrodes 3 and 4 has been neglected since it corresponds to a second-order process. The potential of the voltage electrode μ_3 can thus take any value between μ_1 and μ_2. Since the same holds for μ_4, R_{4pt} can be negative [see Figs. 1(c) and 1(d)].

$$R_{2pt} = \frac{\mu_1 - \mu_2}{I}$$

and

$$R_{4pt} = \frac{\mu_3 - \mu_4}{I},$$

R_{4pt} takes any value between

$$-R_{2pt} \leq R_{4pt} \leq R_{2pt}.$$

2a. Express the chemical potential of the left reservoir (μ_L) of the Hall bar shown on the right in terms of the voltage drop between the left and right lead and μ_R.

To express the chemical potential of the left reservoir (μ_L) in terms of the voltage drop between the left and right lead and the chemical potential of the right reservoir (μ_R), we can use the relationship between chemical potential and voltage.

The chemical potential μ is related to the voltage V by the equation:

$$\mu = -eV$$

where e is the elementary charge.

Given that the voltage drop between the left and right lead is V_{LR}, we can express the chemical potential of the left reservoir (μ_L) as:

$$\mu_L = \mu_R - eV_{LR}$$

Here, μ_R is the chemical potential of the right reservoir, and V_{LR} is the voltage drop from the left to the right lead. This equation assumes that the voltage drop is defined as

$$V_{LR} = V_L - V_R$$

where V_L and V_R are the voltages at the left and right leads, respectively.

So, the chemical potential of the left reservoir is:

$$\mu_L = \mu_R - eV_{LR}$$

2b. A perpendicular magnetic field is applied to the ballistic Hall bar. Sketch the trajectories of electrons when the cyclotron radius is much smaller than the width of the bar. What are the chemical potentials μ_1 and μ_2 and the Hall resistance V_H/I in this case?

Chemical Potentials

In the presence of a magnetic field, the chemical potentials at the edges of the Hall bar will differ due to the Hall effect. If a current I is applied along the length of the bar, a Hall voltage V_H will develop across the width of the bar. The chemical potentials at the two edges (let's say μ_1 and μ_2) will be related to this Hall voltage:

$$\mu_1 = \mu_R + \frac{eV_H}{2} \quad (1)$$

$$\mu_2 = \mu_R - \frac{eV_H}{2} \quad (2)$$

Here, μ_R is the chemical potential of the right reservoir, and V_H is the Hall voltage.

Hall Resistance

The Hall resistance R_H is given by the ratio of the Hall voltage V_H to the applied current I:

$$R_H = \frac{V_H}{I} \quad (3)$$

For a ballistic Hall bar in a strong magnetic field, the Hall resistance is quantized and given by:

$$R_H = \frac{h}{e^2\nu} \quad (4)$$

where h is Planck's constant, e is the electron charge, and ν is the filling factor (an integer in the case of the integer quantum Hall effect).

2c. Sketch the electron trajectories for lower magnetic fields. What happens with the Hall resistance when the field is decreased? Is it possible that the Hall resistance becomes negative (when $B > 0$)?

Hall resistance

As the magnetic field decreases, the Hall resistance $R_H = \frac{V_H}{I}$ typically decreases. This is because the Hall voltage V_H is proportional to the magnetic field B (from the relation $V_H = \frac{BI}{ned}$, where n is the carrier density and d is the thickness of the bar). Thus, reducing B reduces V_H, leading to a lower R_H.

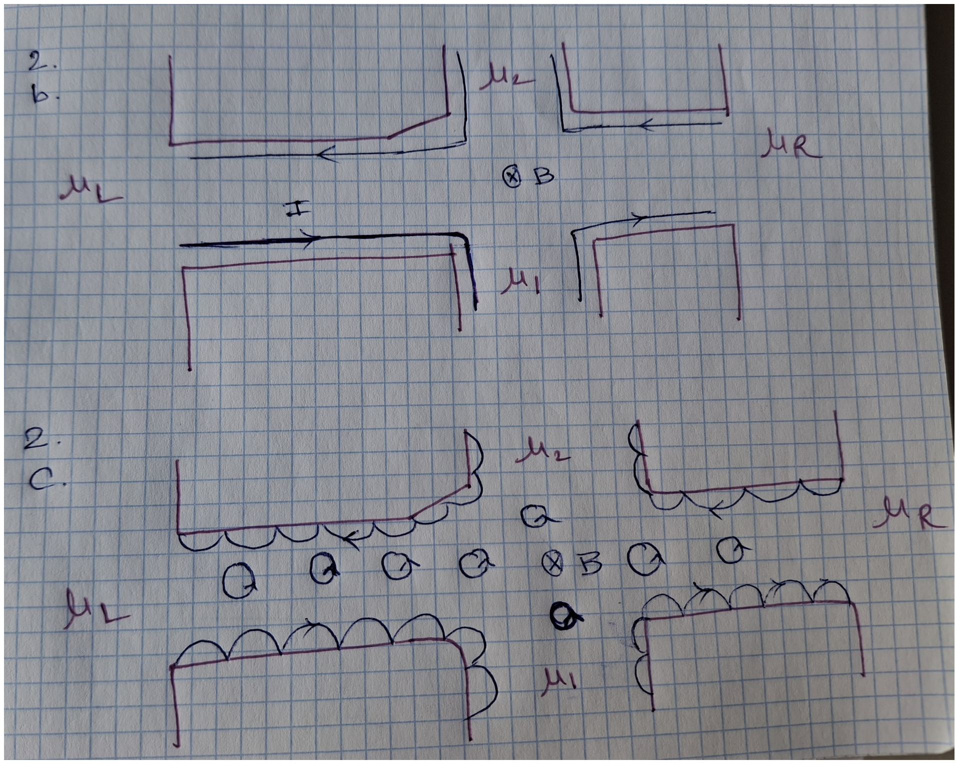

Figure 2: 2(b) Sketch the trajectories of electrons when the cyclotron radius is much smaller than the width of the bar. 2(c) Sketch the electron trajectories for lower magnetic fields..

Possibility of Negative Hall Resistance

In certain materials or under specific conditions (e.g., in systems with both electrons and holes, or in materials with complex band structures), it is possible to observe a negative Hall resistance even when $B > 0$. This can happen if the dominant charge carriers change from electrons to holes, or due to anomalous Hall effects in magnetic materials.

Anomalous Hall Effect

In ferromagnetic or topological materials, the Hall resistance can have contributions from the material's intrinsic magnetization or Berry curvature, leading to a Hall voltage that is not simply proportional to B. In such cases, the Hall resistance can become negative even for $B > 0$.

3. A one dimensional conductor contains a single scatterer with transmission t and an electron with wavefunction $\psi(x)$ = exp(ikx) is sent into the wire.
a. The probability current J determines how fast the probability of finding the electron in the right reservoir changes and is given by:

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) \tag{5}$$

Express the probability current in terms of t and the velocity of the electron.

To solve for the probability current J in terms of the transmission coefficient t and the velocity of the electron, let's follow these steps:

Wavefunction Description: - The incident wavefunction is $\psi_{\text{inc}}(x) = e^{ikx}$. - After encountering the scatterer, the transmitted wavefunction is $\psi_{\text{trans}}(x) = te^{ikx}$.

Probability Current Formula: The probability current J is given by:

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right).$$

Calculate the Derivatives: - For the transmitted wavefunction $\psi_{\text{trans}}(x) = te^{ikx}$:

$$\frac{\partial \psi_{\text{trans}}}{\partial x} = ikte^{ikx}.$$

$$\frac{\partial \psi^*_{\text{trans}}}{\partial x} = -ikt^* e^{-ikx}.$$

Substitute into the Probability Current Formula:

$$J = \frac{\hbar}{2mi}\left(t^* e^{-ikx} \cdot ikte^{ikx} - te^{ikx} \cdot (-ikt^* e^{-ikx})\right).$$

Simplifying inside the parentheses:

$$J = \frac{\hbar}{2mi}\left(ik|t|^2 + ik|t|^2\right) = \frac{\hbar}{2mi} \cdot 2ik|t|^2.$$

$$J = \frac{\hbar k|t|^2}{m}.$$

Express in Terms of Velocity: The velocity v of the electron is related to the wave number k by $v = \frac{\hbar k}{m}$. Therefore:

$$J = v|t|^2.$$

3b. What is the density of states of a one dimensional conductor? How many electrons flow in the channel from the left reservoir when a small voltage V is applied between the left and right reservoir?

Density of States in a One-Dimensional Conductor

The **density of states (DOS)** in a 1D conductor describes the number of available quantum states per unit energy per unit length. For a 1D system, the DOS $g(E)$ is given by:

$$g(E) = \frac{1}{\pi}\sqrt{\frac{2m}{\hbar^2 E}},$$

where:

- m is the effective mass of the electron,
- $\hbar$ is the reduced Planck's constant,
- E is the energy of the electron.

Number of Electrons Flowing in the Channel

When a small voltage V is applied between the left and right reservoirs, electrons flow from the left reservoir to the right reservoir. The number of electrons flowing in the channel can be calculated as follows:

1. Fermi-Dirac Distribution

- The left reservoir has a Fermi energy E_F, and the right reservoir has a Fermi energy $E_F - eV$, where e is the electron charge.
- The probability of an electron occupying a state at energy E is given by the Fermi-Dirac distribution:

$$f(E) = \frac{1}{1 + e^{(E-E_F)/k_BT}},$$

where k_B is the Boltzmann constant and T is the temperature.

2. Number of States

- The number of available states in the energy range dE is $g(E)\,dE$.
- The number of electrons in this energy range is $f(E)\,g(E)\,dE$.

3. Current Carried by Electrons

- The current I is proportional to the number of electrons flowing through the channel. For a small voltage V, the number of electrons flowing from the left reservoir is:

$$N = \int_{E_F - eV}^{E_F} g(E)\,f(E)\,dE.$$

- At low temperatures ($T \to 0$), the Fermi-Dirac distribution becomes a step function, and the integral simplifies to:

$$N = g(E_F)\, eV,$$

where $g(E_F)$ is the density of states at the Fermi energy.

4. Final Expression

- Substituting the 1D density of states $g(E_F) = \frac{1}{\pi}\sqrt{\frac{2m}{\hbar^2 E_F}}$, the number of electrons flowing in the channel is:

$$N = \frac{1}{\pi}\sqrt{\frac{2m}{\hbar^2 E_F}}\, eV$$

3c. Calculate the current through the wire. Does it depend on the electron velocity?

Current Through the Wire

The **probability current** J is given by:

$$J = v|t|^2,$$

where:

- v is the electron velocity,
- t is the transmission coefficient.

The **electric current** I is related to the probability current J and the electron charge e:

$$I = eJ.$$

Substituting the expression for J:

$$I = ev|t|^2.$$

Dependence on Electron Velocity

The current I depends on:

- The electron velocity v,
- The transmission coefficient t,

- The electron charge e.

Thus, the current through the wire **does depend on the electron velocity** v. Specifically, the current is directly proportional to the electron velocity.

3d. When the width of the wire is larger than λ_F, more than one conduction channel is open. For each channel, the transmission can be different. Show that the conductance of the wire is given by:

$$G = \frac{2e^2}{\hbar} \sum_n |t_n|^2. \tag{6}$$

When the width of the wire is larger than the Fermi wavelength λ_F, the electron wavefunctions can form standing waves across the width of the wire. These standing waves correspond to different transverse modes or conduction channels. Each channel n has its own transmission coefficient T_n.

Current in Each Channel

For each conduction channel n, the current I_n can be expressed using the Landauer formula:

$$I_n = \frac{2e}{h} \int T_n(E)(f_L(E) - f_R(E))\, dE,$$

where:

- $T_n(E) = |t_n(E)|^2$ is the transmission probability for channel n at energy E,
- $f_L(E)$ and $f_R(E)$ are the Fermi-Dirac distribution functions in the left and right reservoirs, respectively,
- The factor of 2 accounts for spin degeneracy.

Total Current

The total current I through the wire is the sum of the currents through all individual channels:

$$I = \sum_n I_n = \frac{2e}{h} \sum_n \int T_n(E)(f_L(E) - f_R(E))\, dE.$$

Conductance at Zero Temperature

At zero temperature, the Fermi-Dirac distribution functions become step functions, and the integral simplifies. For a small applied voltage V, the difference in the Fermi functions $f_L(E) - f_R(E)$ is non-zero only in a small energy range around the Fermi energy E_F. The conductance G is then given by:

$$G = \frac{I}{V} = \frac{2e}{h} \sum_n T_n(E_F).$$

Since $T_n(E_F) = |t_n(E_F)|^2$, we can write:

$$G = \frac{2e^2}{h} \sum_n |t_n(E_F)|^2.$$

3e. Assume that the scatterer is a rectangular potential of height $V_0 > E_F$ and size d. Calculate the (energy-dependent) transmission coefficient t(E) and the make a plot of the voltage dependence of the differential conductance.

1 Schrödinger Equation in Different Regions

Consider a potential barrier defined as:

$$V(x) = \begin{cases} 0, & x < 0 \quad \text{(Region I)} \\ V_0, & 0 \le x \le d \quad \text{(Region II)} \\ 0, & x > d \quad \text{(Region III)} \end{cases} \tag{7}$$

The time-independent Schrödinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E_F\psi(x). \tag{8}$$

2 Wavefunctions in Each Region

Region I ($x < 0$):

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}, \tag{9}$$

where $k = \frac{\sqrt{2mE_F}}{\hbar}$.

Region II ($0 \le x \le d$):

$$\psi_{II}(x) = Ce^{\kappa x} + De^{-\kappa x}, \tag{10}$$

where $\kappa = \frac{\sqrt{2m(V_0 - E_F)}}{\hbar}$.

Region III ($x > d$):

$$\psi_{III}(x) = Fe^{ikx}, \tag{11}$$

assuming no reflected wave in this region.

3 Boundary Conditions

The wavefunction and its derivative must be continuous at $x = 0$ and $x = d$:

$$\psi_I(0) = \psi_{II}(0), \quad \psi'_I(0) = \psi'_{II}(0), \tag{12}$$

$$\psi_{II}(d) = \psi_{III}(d), \quad \psi'_{II}(d) = \psi'_{III}(d). \tag{13}$$

Applying the boundary conditions at $x = 0$:

$$A + B = C + D, \tag{14}$$

$$ik(A - B) = \kappa(C - D). \tag{15}$$

Applying the boundary conditions at $x = d$:

$$Ce^{\kappa d} + De^{-\kappa d} = Fe^{ikd}, \tag{16}$$

$$\kappa(Ce^{\kappa d} - De^{-\kappa d}) = ikFe^{ikd}. \tag{17}$$

4 Transmission Coefficient $t(E_F)$

The transmission coefficient $t(E_F)$ is defined as:

$$t(E_F) = \frac{F}{A}. \tag{18}$$

By solving the system of equations obtained from the boundary conditions, we get:

$$t(E_F) = \frac{2ik\kappa e^{-ikd}}{(k^2 - \kappa^2)\sinh(\kappa d) + 2ik\kappa\cosh(\kappa d)}. \tag{19}$$

Differential Conductance

The differential conductance G is defined as the derivative of the current I with respect to the voltage V:

$$G = \frac{dI}{dV}.$$

Substituting the expression for I:

$$G = \frac{d}{dV}\left(\frac{2e}{h}\int_{E_F}^{E_F+eV} |t(E)|^2 dE\right).$$

Using the Leibniz rule for differentiation under the integral sign:

$$G = \frac{2e}{h}|t(E_F + eV)|^2 \cdot e.$$

Simplifying, we get:

$$G = \frac{2e^2}{h}|t(E_F + eV)|^2.$$

Magnitude Squared of $t(E_F + eV)$

To find $|t(E_F + eV)|^2$, we take the magnitude squared of the complex expression:

$$|t(E_F + eV)|^2 = \left| \frac{2ik\kappa e^{-ikd}}{(k^2 - \kappa^2)\sinh(\kappa d) + 2ik\kappa\cosh(\kappa d)} \right|^2 .$$

This simplifies to:

$$|t(E_F + eV)|^2 = \frac{4k^2\kappa^2}{|(k^2 - \kappa^2)\sinh(\kappa d) + 2ik\kappa\cosh(\kappa d)|^2}.$$

Denominator Calculation

The denominator is:

$$\left|(k^2 - \kappa^2)\sinh(\kappa d) + 2ik\kappa\cosh(\kappa d)\right|^2 = \left[(k^2 - \kappa^2)\sinh(\kappa d)\right]^2 + [2k\kappa\cosh(\kappa d)]^2 .$$

$$|t(E_F + eV)|^2 = \frac{4k^2\kappa^2}{[(k^2 - \kappa^2)\sinh(\kappa d)]^2 + [2k\kappa\cosh(\kappa d)]^2}. \tag{20}$$

Substituting this into the formula for G:

$$G = \frac{2e^2}{h} \cdot \frac{4k^2\kappa^2}{[(k^2 - \kappa^2)\sinh(\kappa d)]^2 + [2k\kappa\cosh(\kappa d)]^2}. \tag{21}$$

Thus, the final expression for the conductance is:

$$G = \frac{8e^2}{h} \cdot \frac{k^2\kappa^2}{(k^2 - \kappa^2)^2\sinh^2(\kappa d) + 4k^2\kappa^2\cosh^2(\kappa d)}. \tag{22}$$

Thus,

$$k = \frac{\hbar}{\sqrt{2m(E_F + eV)}}$$

$$\kappa = \frac{\sqrt{2m(V_0 - (E_F + eV))}}{\hbar}$$

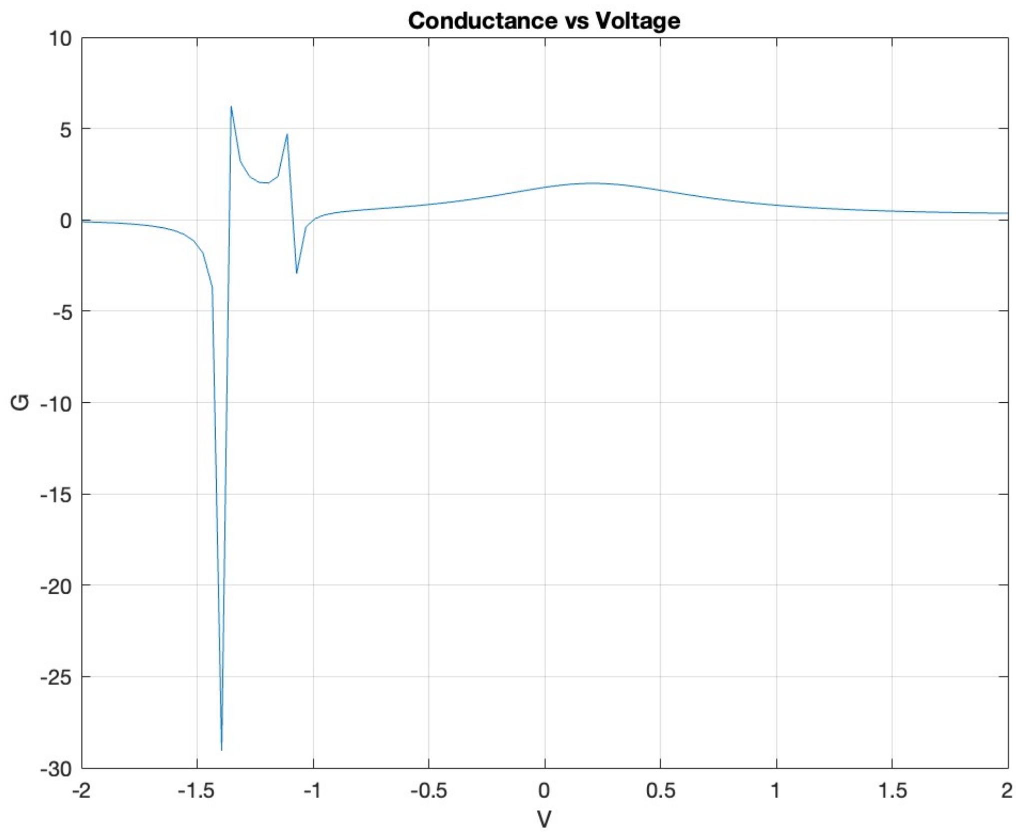

Figure 3: Conductance vs voltage plot.