# 2.2.5 Catastrophes

Nonlinear dynamics, chaos and complex systems

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This section is based on Exercise 3.7.4 of (S. H. Strogatz, *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering*, 2nd ed., Westview Press, 2018). As we have started to point out through previous examples, as parameters change, jumps may occur, which can be catastrophic for the equilibrium, for instance that of a bridge or ecological systems. Let us see an example of this phenomenon. Previously, we introduced a model to describe the harvesting of a population

$$\dot{x} = x - x^2 - h, \tag{1}$$

where $h$ is a parameter. This model states that the harvesting is independent of the population and it allows $x$ to be negative. We note that $x^* = 0$ is not a fixed point for all $h$. Let us consider a fish population. When fewer fish are available, it is harder to find them, and therefore the daily catch drops. A refinement of Eq. (1) is

$$\dot{x} = rx\left(1 - \frac{x}{N}\right) - \frac{Hx}{A+x}, \tag{2}$$

with first part on the right side of the equation being the logistic model and the second part representing the drop in daily catch when $x$ is small, tending to $M$ for large $x$. For a physical system, $r, H, A, x > 0$. $x^* = 0$ is an equilibrium point. What are the dynamics described by this model?

First, we rewrite the model with dimensionless variables in order to deduce the number of parameters. We rescale $x \to x/N$, $t \to tr$, and define $a = A/N$, $h = H/rN$. The model can be recast into:

$$\dot{x} = x(1-x) - \frac{hx}{a+x}, \tag{3}$$

From a simple analysis, we can see that the solutions depend only on a and h. Let us first look at the number of fixed points, depending on $a$ and $h$. $x^* = 0$ is always a fixed point. Other fixed points are the roots of the equation:

$$x^2 + (a-1)x + (h-a) = 0 \tag{4}$$

Investigating the discriminant $\Delta = (a+1)^2 - 4h$,

- $\Delta < 0$, $h > \frac{1}{4}(a+1)^2$, there are no solutions and $x^*$ is the only fixed point. In the $(a, h)$ parameter space shown in Fig. 1, this domain corresponds to regionI;
- $\Delta = 0$, $h = \frac{1}{4}(a+1)^2$, then $x^* = \frac{1-a}{2}$ for $0 < a < 1$, and $x^* = 0$ are fixed points;
- $\Delta > 0$, there are two real roots $x_1$ and $x_2$. Eq. (4) is $x^2 - (x_1 + x_2)x + x_1 x_2 = 0$. Let's distinguish the options depending on the sign of $x_1$ and $x_2$:
  a) If $x_1, x_2 > 0$, then $a - 1 < 0, h - a > 0$, meaning that $\Delta > 0, a < 1, a > h$. $x_1$ and $x_2$ are fixed point in addition to $x^* = 0$ (region II in Fig. 1);
  b) If $x_1, x_2 < 0$, then $a - 1 > 0, h - a > 0$, hence $\Delta > 0, 1 < a < h$, and only $x^* = 0$ is a fixed point (region III in Fig. 1);
  c) $x_1 > 0$, $x_2 < 0$, then $h - a < 0$, hence $\Delta > 0, h < a$, and we have one fixed point in addition to $x^* = 0$ (region IV in Fig. 1).

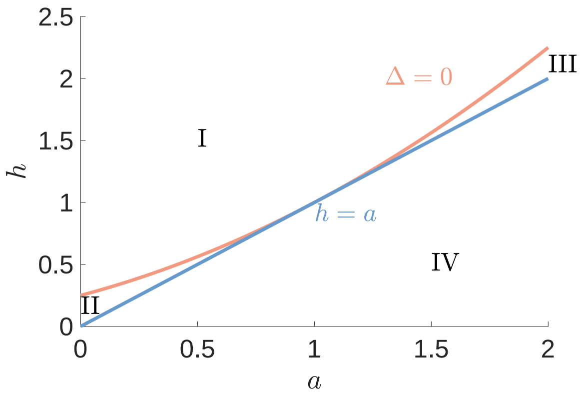


Figure 1: Parameter space $(a, h)$ for fixed points of system in Eq. (4).

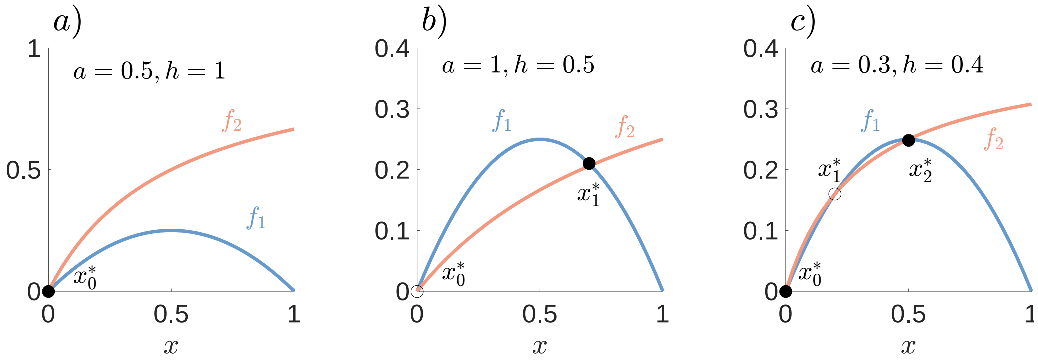


Figure 2: Fixed points stability for system in Eq. (4).

Considering the stability of the fixed points, the derivative of Eq. (4) is

$$f^{'}(x) = 1 - 2x - \frac{ha}{(a+x)^2}. \tag{5}$$

For $x^* = 0$, $f^{'}(0) = (a - h)/a$, therefore $x^*$ is stable for $h > a$ and unstable for $h < a$.

For $x_1$ and $x_2$, it is more easier to investigate the phase portrait. Considering that $\dot{x} = f_1 - f_2$ with $f_1 = x(1 - x)$ and $f_2 = hx/(a + x)$, the fixed points are given by $f_1(x^*) = f_2(x^*)$, and $x^*$ is stable if $f_1^{'}(x^*) < f_2^{'}(x^*)$. The cases are depicted in Fig. 2 $a$), $b$), and $c$).

(a) In case $a$), corresponding to region I, $x^*$ is the only stable fixed point, fishing is unsustainable.

(b) In case $b$), corresponding to region IV, $x^* = 0$ is an unstable fixed point, and $x_1^* > 0$ is a stable fixed point. This condition corresponds to the sustainability of fishing.

(c) Finally, in case $c$), corresponding to region II, $x_0^*$ is stable, $x_1^*$ is unstable, and $x_2^* > x_1^*$ is stable. For this case, a possible dangerous situation related to the stability of $x_0^*$ may arise.

Is there the possibility of catastrophic events? Let us fix $a < 1$, and let us look at the bifurcation diagram as a function of h as shown in Fig. 3. If $h$ exceeds a fixed value, then a jump to the $x_0^* = 0$ equilibrium occurs and even by reducing $h$ one cannot recover a finite population.

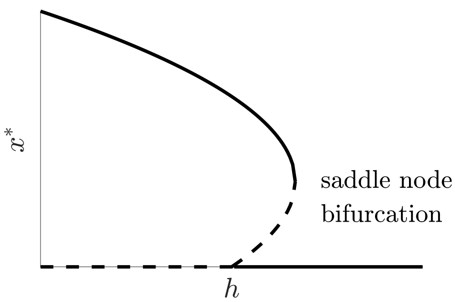


Figure 3: Bifurcation diagram in $h$ for system in Eq. (4) with $a < 1$.