3 Renormalization group theory

3.1 *Introduction*

This section is an elaboration of some ideas raised in §3.4, so a review of that section would make a good foundation to the reading of this one. Note, in particular, Table 3.1 and Fig. 3.6. For new results first note that in fact not only is the nature of period doubling universal but so, in a sense to be seen soon, is the order of the p-cycles which arise at the bifurcations as the parameter increases. Thus Fig. 3.6 has the same *qualitative* appearance for a wide class of maps of which the logistic map is just one. This follows from a remarkable theorem.

Šarkovskii's theorem. If $F: \mathbb{R} \to \mathbb{R}$ is continuous, F has a k-cycle and $l \lhd k$ in the following ordering of all the positive integers, then F also has an l-cycle:

$$1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd 2^4 \lhd \cdots$$

$$\cdots$$

$$\cdots \lhd 2^3 \cdot 9 \lhd 2^3 \cdot 7 \lhd 2^3 \cdot 5 \lhd 2^3 \cdot 3$$

$$\cdots \lhd 2^2 \cdot 9 \lhd 2^2 \cdot 7 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 3$$

$$\cdots \lhd 2 \cdot 9 \lhd 2 \cdot 7 \lhd 2 \cdot 5 \lhd 2 \cdot 3$$

$$\cdots \lhd 9 \lhd 7 \lhd 5 \lhd 3.$$

This powerful theorem with so few hypotheses is due to Šarkovskii (1964); a simpler proof related by Devaney (1989) is more accessible. The theorem is valid only for one-dimensional maps. The converse of the theorem is in fact also true, i.e. if $l \lhd k$ then there exists a continuous function $F: \mathbb{R} \to \mathbb{R}$ such that F has a cycle of period l but not one of period k.

Note that first the powers of 2 are listed in ascending order, then the products of the powers of 2 (in descending order) and the odd numbers (in descending order). The theorem means, for example, that if F has a 10-cycle then it also has a 176-cycle, because $176 = 2^4 \cdot 11 \lhd 2 \cdot 5 = 10$. In particular, it implies that if F has a k-cycle where k is not a power of 2 then F has an infinity of cycles, and if F has a finite number of cycles then all their periods are powers of 2. It also has the following corollary.

Corollary. If F has a three-cycle then F has an l-cycle for all positive integers l.

This astonishing corollary has been epitomized as 'period three implies chaos' (Li & Yorke 1975). To understand the background to the epitome, recall that, although the theorem tells nothing about the stability of the l-cycles, experience of the logistic map in §3.4 suggests that almost all if not all the l-cycles will be unstable. So the cycles will play the role of the repellers in the metaphor of the pin-ball machine. Also recall that the logistic map $F(a,x) = ax(1-x)$ has stable cycles in the 'windows' of its parameter a. For example, it can be seen in Fig. 3.5(b) that F has two stable six-cycles, first a six-cycle on its own account and secondly a six-cycle from the period doubling of the three-cycle. The six-cycle is visited by $F^n(a,x)$ in different orders in each of the two cases. Šarkovskii's theorem does not cover the multiplicity of a cycle of a given period, so it does not imply a universal order of the appearance of cycles at the bifurcations of a difference equation as a parameter increases. The theorem *suggests* period doubling of a k-cycle to $2^n \cdot k$-cycles for $k = 3, 5, \ldots$ as well as 2; in fact each of these sequences of period doublings leads to chaos with a Feigenbaum relation of the form (3.4.9) but with a different universal constant δ for each value of k. Again, it is possible that only a finite sequence of flip bifurcations occurs as a parameter increases, in which case there is no route to chaos by period doubling.

Now we move on to examine the detailed structure of period doubling. It is a good example of self-similarity. Period doubling is found to be characterized by a universal scale α for the state variable x as well as the scale δ for the parameter a. The structure of the period doubling is therefore revealed by *renormalization*, the name being used for 40 years by theoretical physicists to describe groups of scaling transformations in the theories of particle physics and of phase transitions. To explain renormalization group theory, we shall first introduce the concept of *superstability* and then the scales themselves.

Numerical calculations of the value a_r of a at which a 2^r-cycle arises from a flip bifurcation are especially difficult, because the cycle is very weakly stable when a is near to a_r and so computations over a long time are needed to calculate the eigenvalue accurately. However, calculations of the value A_r of a at which the 2^r-cycle is *most* stable are much easier. Accordingly we say that a cycle is *superstable* if it is as linearly stable as it can be, e.g. if the eigenvalue of F^{2^r} is $q = 0$ at each point of the 2^r-cycle

$\{X_1, X_2, \ldots, X_{2^r}\}$. For an example of superstability with a cycle of period one, the Newton–Raphson method to calculate a fixed point is superstable and so converges very rapidly once a close approximation to a fixed point has been found.

Example 4.3: the logistic map. It is shown in §3.4 that if $F(a, x) = ax(1 - x)$ then there is a stable fixed point, i.e. a 2^0-cycle, $X = (a - 1)/a$ for $1 < a \leqslant a_1 = 3$ with eigenvalue $q = 2 - a$. Therefore $A_0 = 2$ because $q = 0$ when $a = 2$; then $X = \frac{1}{2}$. Also there is a stable 2^1-cycle $\{X_1, X_2\}$ when $a_1 < a \leqslant a_2 = 1 + \sqrt{6}$ with $q = [\partial F^2(a,x)/\partial x]_{X_1} = F_x(a, X_1)F_x(a, X_2) = 4 + 2a - a^2$. Therefore A_1 is the zero of q such that $a_1 < A_1 < a_2$, i.e. $A_1 = 1 + \sqrt{5} = 3.236$; then $X_1 = \frac{1}{2}, X_2 = \frac{1}{4}(1 + \sqrt{5})$. □

Recall that, by use of the chain rule, the multiplier q determining the stability of the 2^r-cycle can be shown to have the same value $\prod_{j=1}^{2^r} F_x(a, X_j)$ at each point of the 2^r-cycle, so that $q = 0$ if and only if the derivative of F vanishes at one point of the cycle. Therefore, if F is a smooth convex function with a simple maximum at X_m then $q = 0$ if and only if $X_j = X_m$ for one value of j, i.e. if and only if X_m belongs to the 2^r-cycle, i.e.

$$F^{2^r}(A_r, X_m) = X_m.$$

Thus A_r is a value such that

$$F_x^{2^r}(A_r, X_m) = 0.$$

In fact if F is a smooth convex function then A_r is the unique value; indeed, as a increases from a_r to A_r to a_{r+1}, $F_x^{2^r}(a, X_1)$ decreases monotonically from 1 to 0 to -1.

We have established that if $a = A_r$ then X_m belongs to the 2^r-cycle. So the other points are $F^j(A_r, X_m)$ for $j = 1, 2, \ldots, 2^r - 1$. Of these points the closest to X_m is $F^{2^{r-1}}(A_r, X_m)$. To see why this is true, first note that each member of a 2^{r-1}-cycle of F is a fixed point of $F^{2^{r-1}}$ and the 2^r-cycle of F contains two 2^{r-1}-cycles of F. The 2^r-cycle of F bifurcates from the 2^{r-1}-cycle of F as a increases through a_r and the line $y = x$ cuts the curve $y = F^{2^{r-1}}(a, x)$ at the two-cycle as well as the fixed point of $F^{2^{r-1}}$ (see, e.g., Fig. 3.4). So, as a increases through a_r, the two points of the two-cycle of $F^{2^{r-1}}$ separate from the fixed point and one another. However, this leaves X_1, the point that becomes X_m when $a = A_r$, closest to $F^{2^{r-1}}(a, X_1)$ and these two points of the 2^r-cycle stay closest as a increases to A_r. To prepare to investigate the scaling of the separations of the points $\{X_1, X_2, \ldots, X_{2^r}\}$ of a 2^r-cycle for large r, define

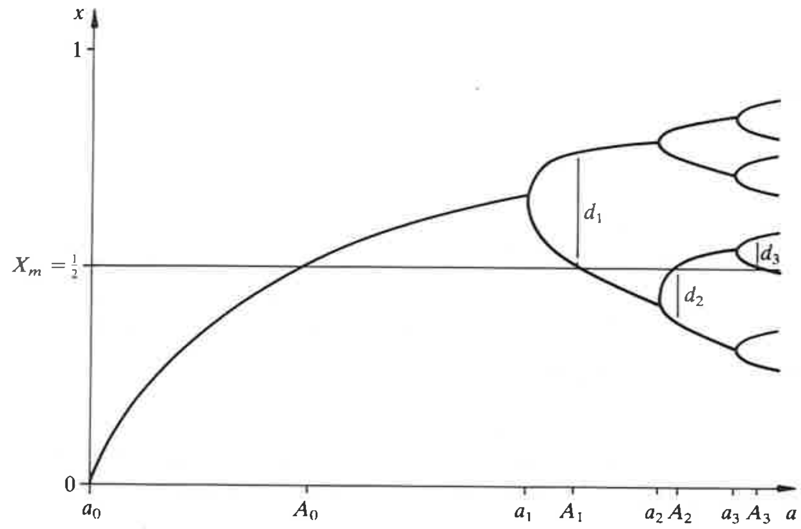

Fig. 4.4 Sketch (not to scale) in the (a, x)-plane of the bifurcation diagram of a one-dimensional map, showing the flip bifurcations and superstable 2^r-cycles.

$$d_r = F^{2^{r-1}}(A_r, X_m) - X_m \qquad \text{for } r = 1, 2, \ldots, \tag{1}$$

the distance from X_m to the nearest other member of the superstable 2^r-cycle. Then the location of the flip bifurcations and superstable cycles is summarized in Fig. 4.4.

3.2 *Feigenbaum's theory of scaling*

We are now ready to describe Feigenbaum's theory of period doubling, although in addition a knowledge of the elements of applied functional analysis will help. It is interesting that Feigenbaum's (1978) paper was rejected by the first journal to which it was submitted (Cvitanović 1984, p. 244). Feigenbaum calculated A_r and d_r numerically for several values of r and for a few functions F and concluded that

$$A_r = a_\infty - B\delta^{-r} + o(\delta^{-r}), \qquad d_r \sim D/(-\alpha)^r \qquad \text{as } r \to \infty, \tag{2}$$

where B, D are constants which depend upon the map F, but $\delta = 4.6692\ldots$ and $\alpha = 2.5029\ldots$ are 'universal' constants which do not. This shows that α is the x-scale of the route to chaos by period doubling much as δ is the a-scale. The scaling of d_r can be expressed as

$$\lim_{r\to\infty} (-\alpha)^r\{F^{2^r}(A_{r+1}, X_m) - X_m\} = -D/\alpha. \tag{3}$$

This leads to the further hypothesis that the limit

$$g_1(x - X_m) = \lim_{r\to\infty} \{g_{1r}(x - X_m)\} \tag{4}$$

exists, where $g_{1r}(x - X_m) = (-\alpha)^r\{F^{2^r}(A_{r+1}, X_m + (x - X_m)/(-\alpha)^r) - X_m\}$, for we see that (3) implies that $g_1(0) = -D/\alpha$, and calculations of $g_{1r}(x - X_m)$ for quite low values of r (rather than infinity) seem to confirm the existence of a limit g_1 independent of F. Then the scaling of $x - X_m$ shows that only the behaviour of F near to its maximum determines g_1 and so it is this behaviour which is responsible for the universality of g_1.

To make the notation a little less cumbersome it is convenient to translate the origin of x to the maximum X_m of F. So henceforth we shall simply put $X_m = 0$ without loss of generality.

Example 4.4: the logistic map. If $F(a, x) = ax(1 - x)$ then we may replace $x - X_m = x - \frac{1}{2}$ by x to get the new function $F(a, x) = a(\frac{1}{4} - x^2)$, ensuring that the maximum of F is now at $x = 0$. Then we find that

$$\begin{aligned} g_{10}(x) &= F(A_1, x) = A_1(\tfrac{1}{4} - x^2), \\ g_{11}(x) &= (-\alpha)F^2(A_2, x/(-\alpha)) \\ &= (-\alpha)A_2\{\tfrac{1}{4} - A_2^2(\tfrac{1}{4} - x^2/\alpha^2)^2\} \\ &= \alpha A_2\{\tfrac{1}{4}(\tfrac{1}{4}A_2^2 - 1) - \tfrac{1}{2}A_2^2x^2/\alpha^2 + A_2^2x^4/\alpha^4\}. \end{aligned}$$

We can similarly find $g_{1r}(x)$ for $r = 2, 3$, etc. and plot the curves $y = g_{1r}(x)$ in the (x, y)-plane to see that a limiting function g_1 seems to emerge as r increases. □

The essence of this scaling of x is captured by the operator T defined by

$$\mathrm{T}\psi(x) = -\alpha\psi(\psi(-x/\alpha)) \tag{5}$$

for all continuous functions ψ. Then

$$\begin{aligned} \mathrm{T}g_1(x) &= -\alpha g_1(g_1(-x/\alpha)) \\ &= -\alpha \lim_{r\to\infty} \{(-\alpha)^r F^{2^r}(A_{r+1}, (-\alpha)^r F^{2^r}(A_{r+1}, x/(-\alpha)^{r+1}))\}. \end{aligned}$$

Now define ϕ by $\phi(y) = (-\alpha)^r F^{2^r}(A_{r+1}, y/(-\alpha)^r)$, so $\phi^2(y) = \phi(\phi(y)) = (-\alpha)^r F^{2^{r+1}}(A_{r+1}, y/(-\alpha)^r)$. Then taking $y = x/(-\alpha)$, we deduce that

$$\begin{aligned} \mathrm{T}g_1(x) &= \lim_{r\to\infty} \{(-\alpha)^{r+1} F^{2^{r+1}}(A_{r+1}, x/(-\alpha)^{r+1})\} \\ &= \lim_{q\to\infty} \{(-\alpha)^q F^{2^q}(A_q, x/(-\alpha)^q)\}, \\ &= g_0(x), \quad \text{say.} \end{aligned}$$

Similarly, it can be shown that

$$\mathrm{T}g_k(x) = g_{k-1}(x) \qquad \text{for } k = 2, 3, \ldots, \tag{6}$$

where g_k is defined by

$$g_k(x) = \lim_{r\to\infty} \{(-\alpha)^r F^{2^r}(A_{r+k}, x/(-\alpha)^r)\}. \tag{7}$$

Taking the limit as $k \to \infty$ in equation (6), we conclude plausibly that there exists a function

$$g(x) = \lim_{k\to\infty} g_k(x) \tag{8}$$

such that

$$\mathrm{T}g = g, \tag{9}$$

i.e. that there exists a 'fixed point' g of the nonlinear functional operator T. The famous equation (9) was discovered in a discussion between Cvitanović and Feigenbaum (1978, p. 46). We shall incidentally show later that the fixed point g is unstable.

Although we in fact know α from numerical solutions of difference equations, its proper status at this stage of the theory is a constant to be determined from equation (9). To find it first note that if $g(x)$ is a solution of equation (9) then so is $\mu g(x/\mu)$ for all $\mu \neq 0$. So we may, by convention, choose a particular value of μ such that

$$g(0) = 1. \tag{10}$$

Then, on putting $x = 0$ into equation (9) and using (5), it follows that

$$\alpha = -1/g(1). \tag{11}$$

Feigenbaum (1979) verified numerically the above scaling structure, and sought to find g as an even function by expanding $g(x)$ as a series in powers of x^2, truncating the series, and equating coefficients of successive powers of x^2 in equation (9). In this way he found that

$$g(x) = 1 - 1.52763x^2 + 0.10482x^4 - 0.02671x^6 + \cdots, \tag{12}$$

and $\alpha = -1/g(1) = 2.5029\ldots$. Thus α appears as a sort of nonlinear eigenvalue of the functional equation (9).

Example 4.5: quadratic approximation to g. To solve equation (9) for all x, where g is an even function and $g(0) = 1$, and then find $\alpha = -1/g(1)$ approximately, assume that $g(x) = 1 + bx^2$ for some constant b and neglect all higher powers of x. Then substitution into equation (9) gives

$$1 + bx^2 = -\alpha[1 + b\{1 + b(-x/\alpha)^2\}^2]$$
$$= -\alpha(1 + b + 2b^2x^2/\alpha^2 + b^3x^4/\alpha^4).$$

Equating coefficients of x^0 and x^2, and neglecting the term in x^4, we find that

$$1 = -\alpha(1 + b), \qquad b = -2b^2/\alpha.$$

Therefore $\alpha = -1/(1 + b) = -1/(1 - \frac{1}{2}\alpha)$. This gives a quadratic equation for α with solution $\alpha = 1 \pm \sqrt{3}$. But we require $\alpha > 1$. Therefore $\alpha = 1 + \sqrt{3} = 2.73\ldots$, and $b = -\frac{1}{2}\alpha = -1.37\ldots$. It is a crude approximation, but the example shows how to calculate g and α to higher approximations. □

Next we move on to find the scaling of the parameter a in the route to chaos by period doubling, evaluating δ. We shall show that

$$g_k(x) - g(x) \sim \text{constant} \times \delta^{-k}u_1(x) \qquad \text{as } k \to \infty, \tag{13}$$

where δ is the eigenvalue belonging to the first eigenfunction u_1 of the linear operator J_g defined as the Fréchet derivative of the nonlinear operator T evaluated at the 'point' g in the space of continuous functions.

Example 4.6: calculation of the Fréchet derivative of T. The *Fréchet derivative* J_ψ of the operator T 'at' ψ is defined by linearization of T about ψ, i.e. by the equation

$$\mathrm{T}(\psi + \epsilon\phi) = \mathrm{T}\psi + \epsilon \mathrm{J}_\psi\phi + o(\epsilon) \qquad \text{as } \epsilon \to 0 \tag{14}$$

for all (well-behaved) functions ϕ. To find J_ψ we expand

$$\mathrm{T}(\psi + \epsilon\phi) = -\alpha(\psi + \epsilon\phi)(\psi(-x/\alpha) + \epsilon\phi(-x/\alpha))$$
$$= -\alpha\psi(\psi(-x/\alpha) + \epsilon\phi(-x/\alpha)) - \alpha\epsilon\phi(\psi(-x/\alpha) + \epsilon\phi(-x/\alpha))$$
$$= -\alpha\psi(\psi(-x/\alpha)) - \alpha\psi'(\psi(-x/\alpha))\cdot\epsilon\phi(-x/\alpha)) + O(\epsilon^2)$$
$$- \alpha\epsilon\phi(\psi(-x/\alpha)) + O(\epsilon^2) \qquad \text{as } \epsilon \to 0,$$

on assuming that ψ, ϕ are well-behaved and taking a Taylor series,

$$= \mathrm{T}\psi + \epsilon \mathrm{J}_\psi\phi + O(\epsilon^2),$$

where J_ψ is defined by

$$\mathrm{J}_\psi\phi = -\alpha\psi'(\psi(-x/\alpha))\phi(-x/\alpha) - \alpha\phi(\psi(-x/\alpha)). \ □ \tag{15}$$

To proceed to find the a-scaling as $a \to a_\infty$, we expand

$$F(a, x) = F(a_\infty, x) + (a - a_\infty)f(x) + O\{(a - a_\infty)^2\} \qquad \text{as } a \to a_\infty, \tag{16}$$

where $f(x) = F_a(a_\infty, x)$. (Of course, if $F(a, x) = af(x)$ then equation (16) is exact for all a on omission of the remainder term $O\{(a - a_\infty)^2\}$.) Therefore the Taylor expansion of the operator T acting on equation (16) gives

$$\mathrm{T}F(a, x) = -\alpha F(a, F(a, -x/\alpha))$$
$$= -\alpha F(a, F(a_\infty, -x/\alpha) + (a - a_\infty)f(-x/\alpha) + \cdots)$$
$$= -\alpha F(a_\infty, F(a_\infty, -x/\alpha)) - \alpha(a - a_\infty)f(F(a_\infty, -x/\alpha)) + \cdots$$
$$- \alpha(a - a_\infty)f(-x/\alpha)F_x(a_\infty, F(a_\infty, -x/\alpha)) + \cdots$$
$$= \mathrm{T}F(a_\infty, x) + (a - a_\infty)\mathrm{J}_{F(a_\infty, x)}f(x) + O\{(a - a_\infty)^2\} \qquad \text{as } a \to a_\infty.$$

On iteration, this process gives

$$\mathrm{T}^kF(a, x) = \mathrm{T}^kF(a_\infty, x) + (a - a_\infty)\mathrm{J}_{\mathrm{T}^{k-1}F(a_\infty, x)}f(x) + O\{(a - a_\infty)^2\}$$
$$= g(x) + (a - a_\infty)\mathrm{J}_g^k f(x) + O\{(a - a_\infty)^2\} + o(1) \tag{17}$$

as $a \to a_\infty$, $k \to \infty$.

To simplify equation (17) consider the eigenvalue problem

$$\mathrm{J}_g u = \lambda u$$

and suppose that it has eigenvalue λ_j belonging to the eigenfunction u_j for $j = 1, 2, \ldots$, where $\{u_j\}$ is a complete set of continuous functions over the interval on which f is positive, for example a complete set for $C[-\frac{1}{2}, \frac{1}{2}]$ if $F(a, x) = a(\frac{1}{4} - x^2)$. Then

$$f(x) = \sum_{j=1}^{\infty} \xi_j u_j(x)$$

for some constants ξ_j. Therefore

$$\mathrm{J}_g^k f(x) = \sum_{j=1}^{\infty} \xi_j \lambda_j^k u_j(x)$$
$$\sim \xi_1 \lambda_1^k u_1(x) \qquad \text{as } k \to \infty,$$

if we assume that $\xi_1 \neq 0$ and we may take $|\lambda_1| > |\lambda_j|$ for $j = 2, 3, \ldots$. Then let $\delta = \lambda_1$ and $h(x) = \xi_1 u_1(x)$. Therefore equation (17) gives

$$\mathrm{T}^kF(a, x) = g(x) + (a - a_\infty)\delta^k h(x) + o(1) + O\{(a - a_\infty)^2\}$$
$$\text{as } k \to \infty, a \to a_\infty.$$

Therefore

$$T^r F(A_r, 0) - g(0) \sim (A_r - a_\infty)\delta^r h(0) \qquad \text{as } r \to \infty. \tag{18}$$

Now

$$TF(A_r, 0) = -\alpha F(A_r, F(A_r, 0))$$
$$= -\alpha F^2(A_r, 0).$$

On iteration, this gives

$$T^r F(A_r, 0) = (-\alpha)^r F^{2^r}(A_r, 0).$$

But X_m belongs to each superstable cycle, so $F^{2^r}(A_r, X_m) = X_m$, and, after translation of the maximum to the origin, this gives $F^{2^r}(A_r, 0) = 0$. Therefore

$$T^r F(A_r, 0) = 0.$$

Therefore relation (18) gives

$$A_r - a_\infty \sim -g(0)/\delta^r h(0) \qquad \text{as } r \to \infty$$
$$= -\delta^{-r}/h(0),$$

which was anticipated in the first of relations (2).

Feigenbaum (1980) also examined the Fourier spectrum of $\{F^n(x_0)\}$ for 2^r-cycles as $r \to \infty$.

All this, then, is Feigenbaum's heuristic theory of scaling of x and a in the route to chaos by period doubling. The astonishing ubiquity of Feigenbaum's sequence in period doubling of maps of $\mathbb{R}^m$ for $m > 1$, of solutions of differential equations, and of phenomena in laboratory experiments, stems from this theory for one-dimensional maps.

4 Liapounov exponents

In studying chaotic solutions (§§3.3, 3.4) we have met sensitive dependence on initial conditions and met simple examples of neighbouring orbits which separate exponentially. To be more formal we may define an infinite invariant set S of a map $F: \mathbb{R} \to \mathbb{R}$ to *have sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in$ S and all neighbourhoods N (however small) of x there exists $y \in$ N and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set.

It is, moreover, a characteristic of neighbouring chaotic orbits that their separation is an exponential function *on average*, though not necessarily an exact exponential function. It is this rapid separation which makes it impossible in practice to predict the behaviour of a chaotic solution far into the future. This is in contrast to the behaviour of an orbit near an attractor which is a fixed point or a periodic solution. These ideas can be quantified by use of what are called *Liapounov exponents.*

Consider then a continuously differentiable map $F: \mathbb{R} \to \mathbb{R}$ and suppose that there exists λ such that

$$|F^n(x_0 + \epsilon) - F^n(x_0)| \sim \epsilon e^{n\lambda} \qquad \text{as } \epsilon \to 0, n \to \infty$$

provided that $\epsilon e^{n\lambda} \to 0$ also, i.e.

$$\epsilon \left| \frac{dF^n(x_0)}{dx_0} \right| \sim \epsilon e^{n\lambda} \qquad \text{as } n \to \infty,$$

to express the average exponential separation of the orbit starting at $x_0 + \epsilon$ from the orbit starting at x_0. Therefore

$$\lambda = \lim_{N\to\infty} \left\{ \frac{1}{N} \ln \left| \frac{dF^N(x_0)}{dx_0} \right| \right\}$$
$$= \lim_{N\to\infty} \{N^{-1} \ln|F'(x_{N-1})F'(x_{N-2}) \ldots F'(x_0)|\},$$

where $x_n = F^n(x_0)$, on differentiating a function of a function and using induction (cf. Q3.5),

$$= \lim_{N\to\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} \ln|F'(x_n)| \right\}. \tag{1}$$

This shows that λ is a measure of the exponential separation of the neighbouring orbits averaged over all points of an orbit around an attractor.

We now may formally define the *Liapounov exponent* λ of an invariant set of F by the limit (1), if it exists. Sometimes e^λ is called a *Liapounov multiplier* or *Liapounov number.* In general λ depends on the initial point x_0 of the orbit, but it is the same for almost all x_0 in the domain of attraction of a given attractor. We see that for a stable cycle $\lambda < 0$ and neighbouring orbits converge (Q3.5), but that for a chaotic attractor $\lambda > 0$. The Liapounov exponent may be interpreted in terms of information theory as giving the rate of loss of information about the location of the initial point x_0 (Shannon & Weaver 1949) or in terms of Kolmogorov entropy as measuring the disorder of the system (Kolmogorov 1959).

In general λ can only be found by computation, but it can be evaluated analytically in some simple cases.