29 April 2025

Solutions 9 : One-Dimensional Maps

1 Superstable Cycle

(a) Suppose that a well-behaved function f gives rise to a difference system $x_{n+1} = f(x_n)$ that has a p-cycle $\{x_1, \ldots, x_p\}$. The value of

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f(\cdots f(x)))|_{x=x_i} = (f^p)'(x_i)$$

determines the stability of the cycle. Show that $(f^p)'(x_i) = \prod_{j=1}^{p} f'(x_j)$ and therefore it is the same at all points, $1 \leq i \leq p$, of the p-cycle.

There is no prefered starting point of the cycle. Therefore, without loss of generality we can relabel any point x_1. Using the chain rule,

$$\begin{aligned}
(f^p)'(x_1) = (f(f^{p-1}))'(x_1) &= (f^{p-1})'(x_1) \cdot f'(f^{p-1}(x_1)) \\
&= (f^{p-1})'(x_1) \cdot f'(x_p) \\
&= (f^{p-2})'(x_1) \cdot f'(f^{p-2}(x_1)) \cdot f'(x_p) \\
&= (f^{p-2})'(x_1) \cdot f'(x_{p-1}) \cdot f'(x_p) \\
&= \ldots \\
(f^p)'(x_1) &= \prod_{i=1}^{p} f'(x_i)
\end{aligned}$$

The last line is done recursively using $(f^n)'(x_1) = (f^{n-1})'(x_1) \cdot f'(x_n)$ and $(f^0)' = 1$ since f^0 is the identity function. $(f^p)'(x_1)$ is the product of the derivatives of f on all points of the cycle. Relabeling the points does not change this result, therefore this product is the value of $(f^p)'$ on all points x_i of the cycle.

(b) A cycle is said superstable if $(f^p)'(x_i) = 0$. Find the cubic equation that a must satisfy so that $x_{n+1} = f(x_n) = 1 - ax_n^2$ has a superstable 3-cycle.

To have a superstable 3-cycle, we need to find one (out of the three possible) x such that $(f^3)'(x) = 0$ and $x = f^3(x)$. Start with

$$\begin{aligned}
(f^3)'(x) &= f'(f(f(x)))f'(f(x))f'(x) \\
&= (-2af(f(x)))(-2af(x))(-2ax) \\
&= -8a^3 f(f(x))f(x)x
\end{aligned}$$

Note that the $a = 0$ case has $f(x) = 1$ and is not very interesting. Therefore the superstable condition is satisfied if $x = 0$ or $f(x) = 0$ or $f(f(x)) = 0$. This means that either x_1, x_2 or x_3 of the 3-cycle must be zero. Without loss of generality we can relabel and say that $x = x_1 = 0$. Then, the cycle condition is

$$0 = f(f(f(0))) = f(f(1)) = f(1-a) = 1 - a(1-a)^2$$

The values of a to have a superstable cycle can be found by solving the cubic equation. The Mathematica software gives $a = \frac{1}{3}[2 + (\frac{1}{2}(25 - 3\sqrt{69}))^{1/3} + (\frac{1}{2}(25 + 3\sqrt{69}))^{1/3}] \approx 1.7549$ and a complex conjugate pair $a \approx 0.12256 \pm 0.74486\mathrm{i}$.

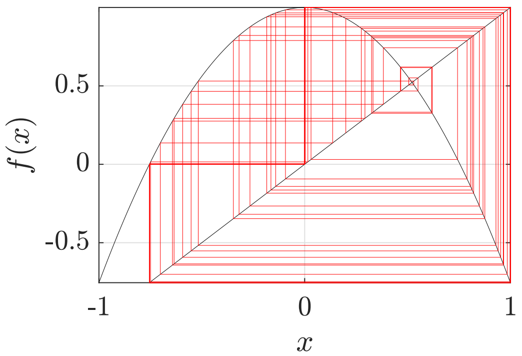
(c) To visualise the superstable cycle found in (b), create a program. Start by computing a few iterations $x_{n+1} = f(x_n)$ and plot x_n as a function of n. Choose the value of a corresponding to the superstable cycle and check that iterations converge to the superstable 3-cycle.

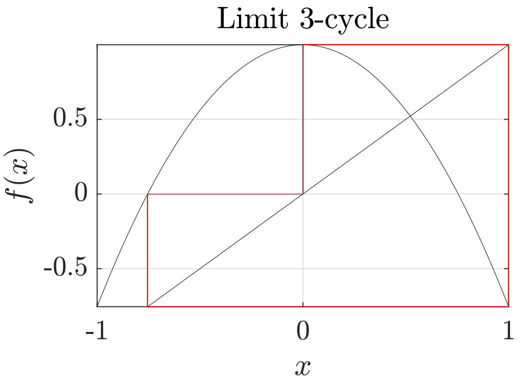
See the code in file `plot_evolution.m`. The `main.m` file uses `roots` to numerically find a, using the fact that $1 - a(1-a)^2 = 0 = -a^3 + 2a^2 - a + 1$. The plots with $x_0 = 0.2$ shows how the system evolves, to finally converge to the limit 3-cycle.

BONUS

(d) Plot the cobweb diagram. This shows both the $y = f(x)$ function and the $y = x$ diagonal. On top of that, for each x_n, the vertical line from $y = x_n$ to $y = f(x_n)$ and the horizontal from $y = f(x_n)$ back to $y = x_n$ are drawn.

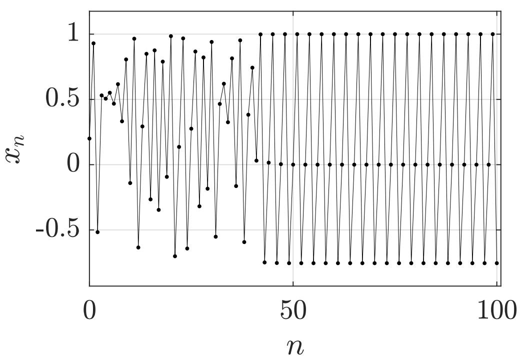
See the code in file `cobweb.m`.

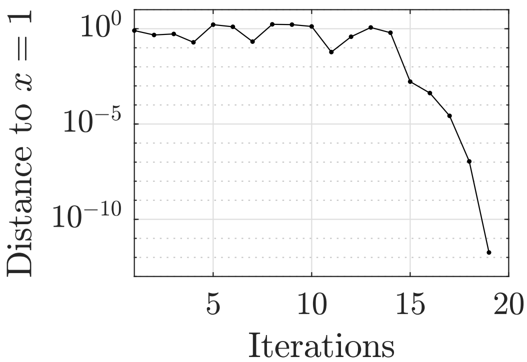




(e) A sequence of one in three iterations, e.g. x_{3k} with $k \in \mathbb{N}$, converges to one of the points on the limit cycle. Make a plot of the distance, in absolute value, between such a sequence and the point on the limit cycle towards which it converges. Describe the superstable convergence.

The code is in `convergence.m`. The distance in logarithmic scale is preferable to study convergence properties. The code plots only iterations x_{3n}, $n \in \mathbb{N}$. Since with $x_0 = 0.2$, it happens that this sequence converges to $x = 1$, we plot $|x_{3n} - 1|$. The system moves around almost chaotically at first, not attracted at all to the limit cycle, up to a certain point where in 6 iterations the distance drops from ≈ 1 to below the precision of the computer, 10^{-15}. The distance is then stuck at precisely zero since the computer cannot go further. The slope plunges down and does not ever reach a constant slope, making it impossible to determine the order of convergence. This is typical of superstable systems.





x_{39}	0.617409991364616
x_{42}	0.001673796282340
x_{45}	0.000421116075103
x_{48}	0.000026845873364
x_{51}	0.000000109344408
x_{54}	0.000000000001814
x_{57}	0

2 Newton's Method

Suppose we want to solve the equation $g(x) = 0$. Newton's method iteratively approximates the roots by evolving the map $x_{n+1} = f(x_n)$ with $f(x) = x - \frac{g(x)}{g'(x)}$, with x_0 an initial guess of the solution.

(a) Prove that under some condition (which you must specify) the fixed points of the Newton map $f(x)$ are the zeros of $g(x)$.

We should evaluate

$$f(x_*) = x_*$$

That is

$$\frac{g(x_*)}{g'(x_*)} = 0$$

Therefore all zeros of g, with non-vanishing derivatives, are fixed points of f (f is not defined where $g' = 0$).

(b) x_* is a root of g and a fixed point of f. What are the stability properties of x_*?

We evaluate

$$f'(x_*) = 1 - \frac{g'(x_*)g'(x_*) - g(x_*)g''(x_*)}{g'(x_*)^2} = 1 - \frac{g'(x_*)^2 - 0}{g'(x_*)^2} = 0$$

The fixed point is superstable. Newton's method converges very rapidly to the fixed point.

(c) Numerically implement Newton's method to solve $0 = 1 - x(1-x)^2$ (we are looking for the solution in the interval $[1, 2]$).

We can reuse the previous codes defining

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{1 - x(1-x)^2}{-3x^2 + 4x - 1}$$

inside `diffEq`. Or it is also possible to use the symbolic language of Matlab, that can analytically compute the derivative $g'(x)$. To declare a symbolic function use `sym f(x);` and to initialise it use `f=1-x*(1-x)∧2`. Then command `diff` can compute the derivative as shown in `Newton_Method.m`. In only 5 iterations the answer is obtained with the maximal precision of 10^{-15}.

x_0	1.500000000000000
x_1	1.857142857142857
x_2	1.764136904761905
x_3	1.754963482395032
x_4	1.754877673713955
x_5	1.754877666246693
x_6	1.754877666246693

(d) In this question, we establish the convergence properties of the Newton method. Prove that, assuming $g'(x)$ always different from zero, and that $g''(x)$ is continuous, the convergence of the Newton's method is quadratic. *Hint* : Start by Taylor-expanding the $g(x)$ near the root x_*, using an explicit form of the second-order remainder.

Taylor-expanding $g(x)$ at x_n and evaluating it at x^*,

$$0 = g(x^*) = g(x_n) + g'(x_n)(x_* - x_n) + R_1,$$

where the Lagrange form of the Taylor series expansion remainder is

$$R_1 = \frac{1}{2}g''(\zeta_n)(x_* - x_n)^2,$$

where ζ_n is in between x_n and x_*. Dividing by $g^{'}(x_n)$ and rearranging gives,

$$\frac{g(x_n)}{g'(x_n)} + (x_* - x_n) = -\frac{g''(\zeta_n)}{2g'(x_n)}(x_* - x_n)^2.$$

Using the definition of x_n we find

$$\eta_{n+1} = \left|\frac{g''(\zeta_n)}{2g^{'}(x_n)}\right|\eta_n^2,$$

having defined $\eta_n = |x_* - x_n|$. Therefore, the order of convergence is at least quadratic if the following conditions are satisfied :
- $g'(x) \neq 0$ for all $x \in I$, where I is the interval $[x_* - |\eta_0|, x_* + |\eta_0|]$;
- $g''(x)$ is continuous for all $x \in I$;
- $M|\eta_0| < 1$

where M is given by

$$M = \frac{1}{2}\left(\sup_{x\in I}|g''(x)|\right)\left(\sup_{x\in I}\frac{1}{|g'(x)|}\right).$$

If there conditions hold,

$$|\eta_{n+1}| \leq M\eta_n^2.$$

While this derivation is rigorous, a similar conclusion can be reached by Taylor-expanding in the neighborhood of x* to higher order, not evaluating the Lagrange reminder then, but assuming it is well behaved

$$\eta_{n+1} = \left|\frac{g''(x_*)}{2g'(x_*)}\right|\eta_n^2 + O(\eta_n^3),$$

thus recovering the familiar form of quadratic convergence. These proofs can be found on Wikipedia's page about the Newton's method on 28/04/2025.

(e) Let us explore the behaviour when $g'(x^*) = 0$. Explore the $g(x) = x^2$ and $g(x) = x^{1/3}$. What happens in these cases ?

The function $g(x) = x^2$ has a root at 0. Since g is continuously differentiable at its root, the theory guarantees that Newton's method as initialized sufficiently close to the root will converge. However, since the derivative g' is zero at the root, quadratic convergence is not ensured by the theory. The Newton iteration is given by $x_{n+1} = x_n/2$, showing that the method can be initialised anywhere and converge to zero with a linear rate (i.e. exponential decay).

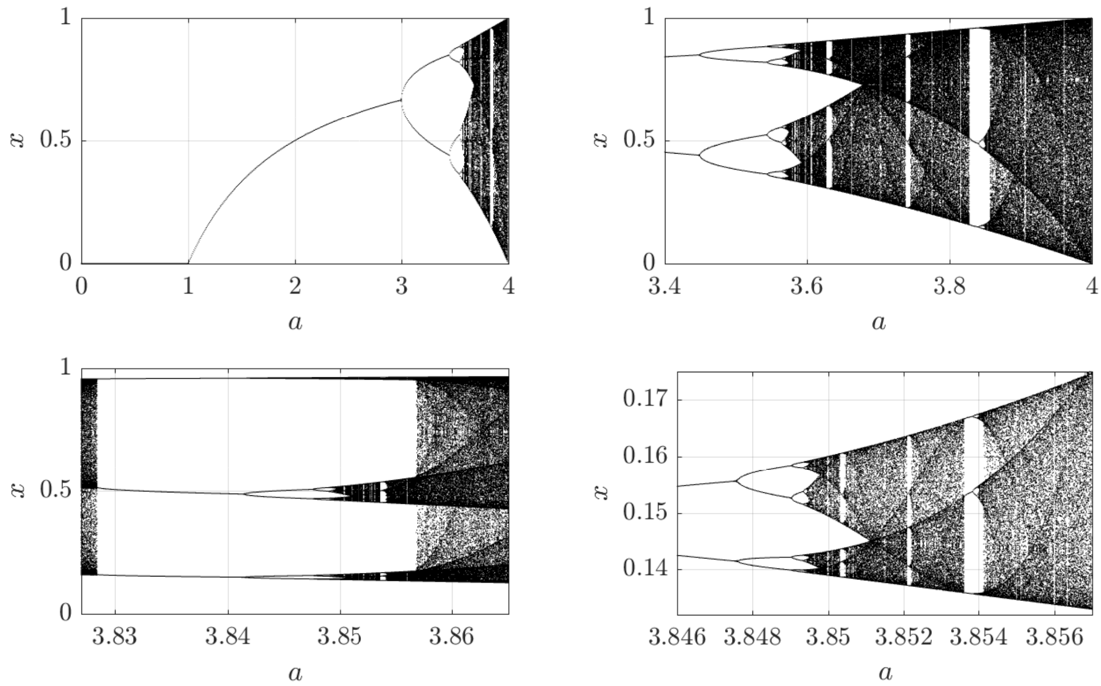
For $g(x) = x^{1/3}$, the map is $x_{n+1} = -2x_n$. Unless the method is initialised at 0, the sequence will fail to converge and the map will oscillate with an exponential divergence. This failure of convergence is not contradicted by the analytic theory, since in this case g is not differentiable at its root.

3 Bifurcation Diagram

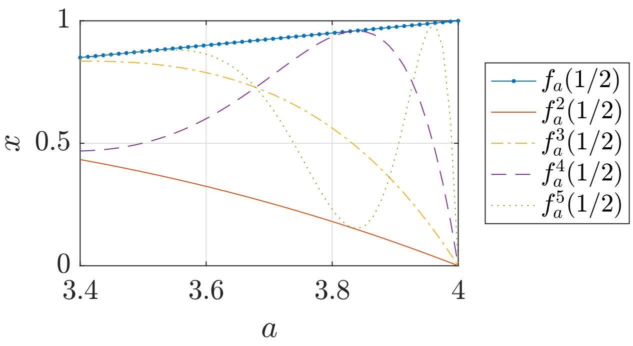
The most studied difference equation is the logistic equation $x_{n+1} = f_a(x_n) = ax_n(1 - x_n)$. What a complicated behaviour it shows, as a is varied from 0 to 4, despite its very simple form !

(a) Make a program that generates the orbit diagram. Discretise the parameter a on a set of values a_i. For each a_i, iterate the map. After a certain number of iterations, the system converges to a fixed point, a limit cycle, or just exhibits chaotic behaviour. Plot the iterations, after the initial transient, on the vertical line $a = a_i$.

The code is in `bifurcation_diagram.m`. At first, with $a \in [0, 1]$ the only fixed point is $x^* = 0$. At $a = 1$ the fixed point moves up, then splits in two at $a \approx 3$, creating a 2-cycle. Taking a closer look at this region we see that the cycle doubles in period to become a 4-cycle at $a \approx 3.45$, then doubles again and again. It cannot be seen with finite precision, but the period doubles an infinite number of times and then the system becomes chaotic at $a \approx 3.57$. Zooming again near $a = 3.83$, we see the system is not chaotic there, there is a 3-cycle. A final zoom on the bottom point of this 3-cycle shows that we get the exact same pattern as before, but upside down. The orbit diagram of a simple quadratic logistic equation hides a complex fractal.



(b) Plot the graphs $(a, f_a^n(1/2))$, for $n \in \{1, 2, 3, 4, 5\}$, on the interval $a \in [3.4, 4]$. Notice how these graphs correspond to the denser regions of the orbit diagram. Try to qualitatively explain why.



The plots are made in the `main.m` file. The high point density comes from the fact that $x = 1/2$ is the maximum of the logistic function $f_a(x)$. There the derivative is flat. It causes all points in the neighborhood of $x = 1/2$ to concentrate near $f_a(1/2) = \frac{a}{4}$ when they are applied f_a. Similarly after two iterations they are concentrated around $f_a^2(1/2)$.

(c) The point $(A, X) \approx (3.7, 0.7)$ is particularly dense on the numerical bifurcation diagram. Taking advantage of the observations in (b), use Mathematica to obtain its analytical coordinates.

This big wedge is the intersection of $f_a^3(1/2)$, $f_a^4(1/2)$ and $f_a^5(1/2)$, it is dense because it superimposes multiple $f_a^n(1/2)$ functions that are dense themselves. It can be found by asking Mathematica to solve $f_A^3(1/2) = f_A^4(1/2)$ and look for the solution $A \approx 3.7$. Then X is $f_A^3(1/2)$. On Mathematica use commands

```
f[a_,x_]:= a x (1 - x)
f3[a_,x_]:= f[a,f[a,f[a,x]]]
f4[a_,x_]:= f[a,f3[a,x]]
Solve[ f3[a,1/2]==f4[a,1/2], a]
```

It has many different solutions, the correct one being $A = \frac{1}{3}[2 + (152 - 24\sqrt{33})^{1/3} + 2(19 + 3\sqrt{33})^{1/3}] \approx 3.679$. Therefore $X = f_A^3(1/2) \approx 0.7282$.

Another way to solve the problem is to notice that $X = f_A^3(1/2) = f_A^4(1/2) = f_A(f_A^3(1/2)) = f_A(X)$. Meaning the X is one of the two fixed points of f_A. To find the fixed points we solve

$$\begin{aligned} f_A(X) = AX(1-X) &= X \\ X(1-X) &= \frac{X}{A} \\ X(1 - \frac{1}{A} - X) &= 0 \end{aligned}$$

Since it is not $X = 0$ then it is $X = 1 - \frac{1}{A}$. Therefore A is a solution of $1 - \frac{1}{A} = f_A^3(1/2)$. This equation is easier to solve, but Mathematica remains good company as it remains an 8^{th} order polynomial equation.