15 April 2025

Solutions 8 : Chaos

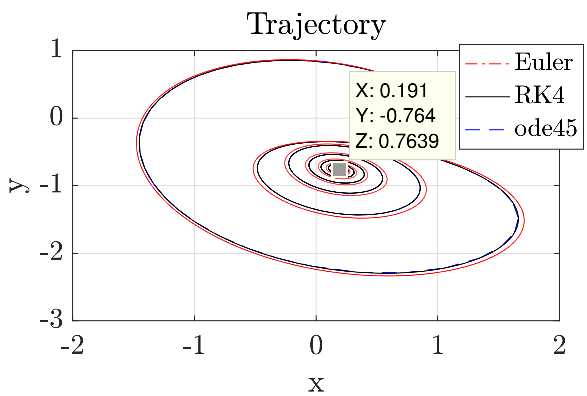
1 The Rössler System

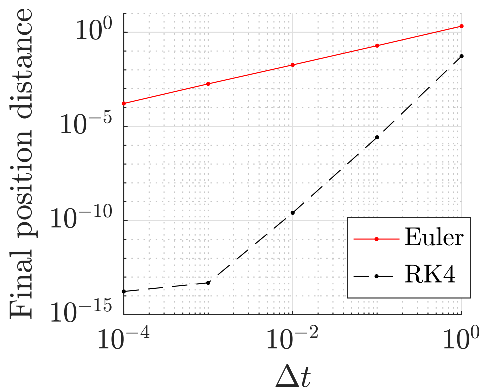
A set of differential equations close to the Lorenz system was studied by the German biochemist Otto Rössler in 1976 to model the dynamics of chemical reactions

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b + x_3(x_1 - c) \end{cases}$$

(a) In Matlab, adapt the code used in the previous weeks for the Lorenz system to integrate and plot the trajectories in the Rössler system using again the forward Euler and Runge-Kutta 4$^{\text{th}}$ order schemes, as well as Matlab's `ode45`. Run a simulation with $a = 1/4$, $b = 1$ and $c = 3/2$ and to verify that the system starting at $\vec{x}_0 = (1, 0, 0)$ converges to a fixed point (using `dt=0.01` should be sufficient).
BONUS : adapt the file `compare_dt.m` to verify that the orders of convergence are the expected ones.

See the attached numerical files for the solution. The code to solve the Rössler system is very similar to that one used for the Lorenz system. Among the changes in `main.m`, we mention that the constants are no longer r , b and σ but a, b and c. Of course, the differential equation in `diffEq.m` is different. Otherwise `integration.m`, `distance.m` and `compare_dt.m` are the same. The trajectory converges to the fixed point $\vec{x}^* \approx (0.19, -0.76, 0.76)$. The orders of convergence, indicated by the slopes on the convergence graph, are the expected ones, that is first order for Euler and fourth order for Runge-Kutta 4.





(b) Identify the nullclines and the fixed points, carefully treating all cases of possible values of the parameters a, b and c. For the rest of the exercise, consider only the $a \neq 0$ case.

The x_1 nullcline is the plane $x_2 = -x_3$. The x_2 nullcline is also a plane, $x_1 = -ax_2$. For the x_3 nullcines there are two possibilities. If $b = 0$, then the x_3 nullcline is given by two intersecting planes $x_3 = 0$ and $x_1 = c$. If $b \neq 0$ it is made up of the two non-intersecting surfaces that satisfy $x_3 = \frac{b}{c-x_1}$. They are both parts of the hyperbolic graph $x_3 = \frac{b}{c-x_1}$, extended at all x_2.
To find the fixed point, we use $x_1 = -ax_2$ to simplify x_1 in the third equation so that $0 = b + x_3(-ax_2 - c)$. Then, use $x_2 = -x_3$ to simplify x_2. This gives $0 = b + x_3(ax_3 - c)$.

- if $a = 0$, obtain a first order polynomial $0 = b - x_3 c$
  - if $c = 0$
    - if $b = 0$ all values of x_3 are acceptable. The fixed points are of the form $(0, -x_3, x_3)$.

- $\cdot$ if $b \neq 0$ there is no fixed point.
  - $\cdot$ if $c \neq 0$ the fixed point is $(0, -b/c, b/c)$
- if $a \neq 0$, it is a second order polynomial $ax_3^2 - cx_3 + b = 0$.
  - $\cdot$ if $c^2 - 4ab < 0$ there are no fixed points.
  - $\cdot$ if $c^2 - 4ab = 0$ there is one fixed point $(\frac{c}{2}, \frac{-c}{2a}, \frac{c}{2a})$
  - $\cdot$ if $c^2 - 4ab > 0$ there are two fixed points $\vec{x}^*_\pm = (\frac{c \pm \sqrt{c^2-4ab}}{2}, \frac{-c \mp \sqrt{c^2-4ab}}{2a}, \frac{c \pm \sqrt{c^2-4ab}}{2a})$

(c) For given parameters a and b, what are the fixed points in the limit of $c \to \infty$?

As $c \to \infty$, we eventually have $c^2 > 4ab$ and, as a consequence, the system presents 2 fixed points. In that limit, we can expand the square root

$$\sqrt{c^2 - 4ab} = c\sqrt{1 - \frac{4ab}{c^2}} \approx c\left(1 + \frac{1}{2}\left(-\frac{4ab}{c^2}\right)\right) = c - \frac{2ab}{c}$$

This means that the first fixed point can be approximated as $(\frac{c+c-2ab/c}{2}, \frac{-c-c+2ab/c}{2a}, \frac{c+c-2ab/c}{2a}) \approx (c, \frac{-c}{a}, \frac{c}{a})$, therefore it diverges to infinity. The other fixed point can be approximated to $(\frac{c-c+2ab/c}{2}, \frac{-c+c-2ab/c}{2a}, \frac{c-c+2ab/c}{2a}) \approx (\frac{ab}{c}, \frac{-b}{c}, \frac{b}{c}) \to (0,0,0)$, therefore converges to the origin.

(d) Identify the Jacobian matrix associated with the linearised system and its characteristic polynomial.

The Jacobian matrix is

$$J(x_1, x_2, x_3) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ x_3 & 0 & x_1 - c \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned}\det[J - \lambda I] &= -\lambda(a-\lambda)(x_1 - c - \lambda) + 1(x_1 - c - \lambda) + x_3(a - \lambda) \\ &= -\lambda^3 + \lambda^2(x_1 - c + a) + \lambda(ac - ax_1 - 1 - x_3) + x_1 - c + ax_3\end{aligned}$$

(e) At the smallest positive value $c = c_{\text{lim}}$ at which fixed points exist, what is the characteristic polynomial? Evaluate the eigenvalues and eigenvectors in the case of $a = 1/4$ and $b = 1$ (feel free to use Mathematica's `Eigenvalue` and `Eigenvector`). You should find a real eigenvalue and two complex conjugate eigenvalues.

At $c = c_{\text{lim}} = \sqrt{4ab} = 1$ the fixed point is $(\frac{c}{2}, \frac{-c}{2a}, \frac{c}{2a})$, the characteristic polynomial becomes

$$\begin{aligned}\det[J - \lambda I] &= -\lambda^3 + \lambda^2(x_1 - c + a) + \lambda(ac - ax_1 - 1 - x_3) + x_1 - c + ax_3 \\ &= -\lambda^3 + \lambda^2(-\frac{c}{2} + a) + \lambda(\frac{ac}{2} - 1 - \frac{c}{2a})\end{aligned}$$

The roots can all be found analytically, $\lambda = 0$ and $\lambda = \frac{1}{2}(p \pm \sqrt{p^2 + 4q})$ with $p = -\frac{c}{2} + a$ and $q = \frac{ac}{2} - 1 - \frac{c}{2a}$. Using values $a = 1/4$ and $b = c = 1$, the Jacobian matrix becomes

$$J\left(\frac{c}{2}, \frac{-c}{2a}, \frac{c}{2a}\right) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c}{2a} & 0 & \frac{c}{2} - c \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \frac{1}{4} & 0 \\ 2 & 0 & -\frac{1}{2} \end{bmatrix}$$

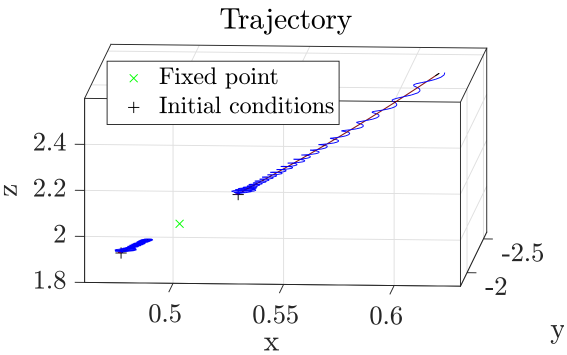
On Mathematica use the following commands

```
J = {{0, -1, -1}, {1, 1/4, 0}, {2, 0, -1/2}};
```

```
Eigenvalues[J]
Eigenvectors[J]
```

It gives the last two eigenvalues $\frac{-1\pm i\sqrt{183}}{8} \approx -0.125\pm1.69i$, which are complex conjugate. An eigenvector for eigenvalue 0 is $v_0 = (1, -4, 4)$, for $\frac{-1+i\sqrt{183}}{8}$ it is $v_+ = (\frac{3+i\sqrt{183}}{16}, \frac{29-i\sqrt{183}}{64}, 1)$ and for $\frac{-1-i\sqrt{183}}{8}$ an eigenvector is $v_- = (\frac{3-i\sqrt{183}}{16}, \frac{29+i\sqrt{183}}{64}, 1)$.

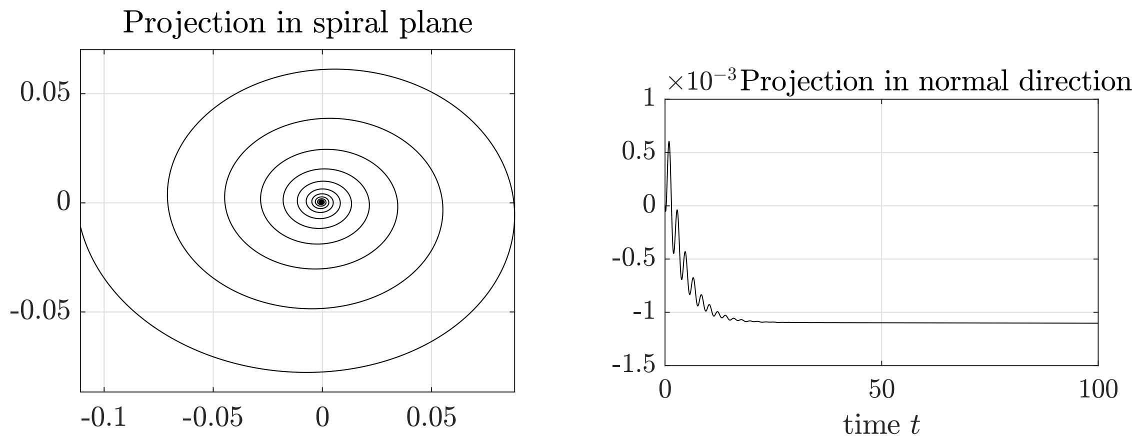
(f) Still with $a = 1/4$, $b = 1$ and $c = c_{\mathrm{lim}}$, compare the numerical solutions to the predictions that linearisation makes in the neighborhood of the fixed point. For the real eigenvalue, the associated eigenspace is a line, passing through the fixed point. Run numerical simulations with initial conditions in the eigenspace, on both sides of the fixed point. Plot and discuss the trajectories. What could be the bifurcation at $c = c_{\mathrm{lim}}$?
Help : Use initial conditions located at a distance of approximately 0.1 from the fixed point. Run the simulation for a time of approximately 100.



See the code in file `check_linearisation.m`. We see that the trajectories follow the direction of the eigenspace associated to eigenvalue 0, which we call E_0. Since the eigenvalue is zero, linearisation is unable to predict what happens in that direction. This also explains why the trajectories evolve slowly in that direction, (just like in the one-dimensional cases of convergence to fixed point with a zero eigenvalue). We can change the final time of the simulation and see that convergence or divergence to the fixed point is not exponential. Indeed, in question (a), with $c = 1.5$, after 2 time units the convergence to the fixed point is clear. Here considerably less progress is made in 100 time units. The trajectory converges towards the fixed point when it starts on the side of E_0 that points roughly towards the origin. (This is the direction along which $\vec{x}^*_-$ moves as c is increased.) That side of the eigenspace is a stable manifold. Conversely, the trajectory diverges from the fixed point when it starts on the other side of E_0. (This is the direction along which $\vec{x}^*_+$ moves as c is increased.) That side is an unstable manifold. The fixed point is therefore half-stable in the E_0 direction. Since it is a fixed point that "appears out of the blue sky" and that separates into two fixed points, we expect it to be a saddle node bifurcation. A close-up analysis of the trajectories shows that they spiral slightly, and that the spirals decay exponentially. This is due to the other eigenvectors of the linearised system that identify a plane where the trajectory shows a stable spiral due to the complex conjugate eigenvalues with a negative real part. This explains why the trajectories stay on E_0 : the perturbations in the other direction decay.

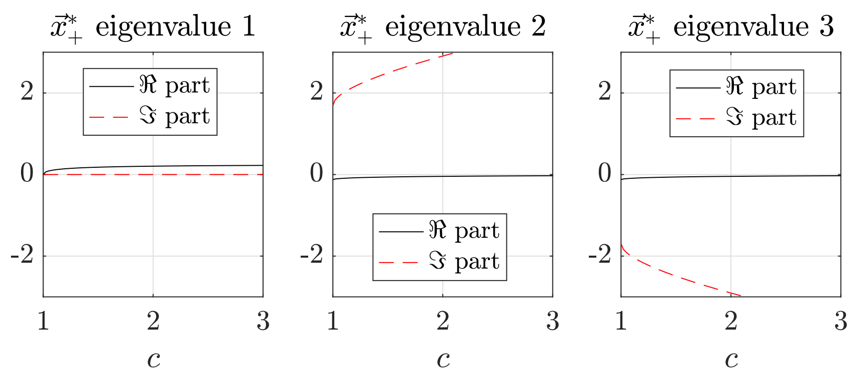
(g) Again with $a = 1/4$, $b = 1$ and $c = c_{\mathrm{lim}}$, verify that the trajectories in the manifold identified by the two complex conjugated eigenvectors are stable spirals, checking that linearisation correctly predicts the behaviour of the trajectories. To do so, look for the eigenvector generating the plane in the following way. Considering the two complex conjugated eigenvectors, $\mathbf{v}$ and $\bar{\mathbf{v}}$, take the plane generators via $\mathbf{u}_1 = Re[\mathbf{v}]$ and $\mathbf{u}_2 = Im[\mathbf{v}]$. The orthogonal direction is given by $\mathbf{u}_1 \times \mathbf{u}_2$. Orthonormalising $\mathbf{u}_1$ and $\mathbf{u}_2$ provides a real basis spanning the spiral plane. Then, project the trajectory on this new basis, and plot separately the spiral plane projected and the normal projection. *Help* : Start the trajectory at a distance of approximately 0.1 from the fixed point. You can use the Matlab function `orth` to get an orthonormal basis of the plane, and `cross` to get the normal direction of the plane.

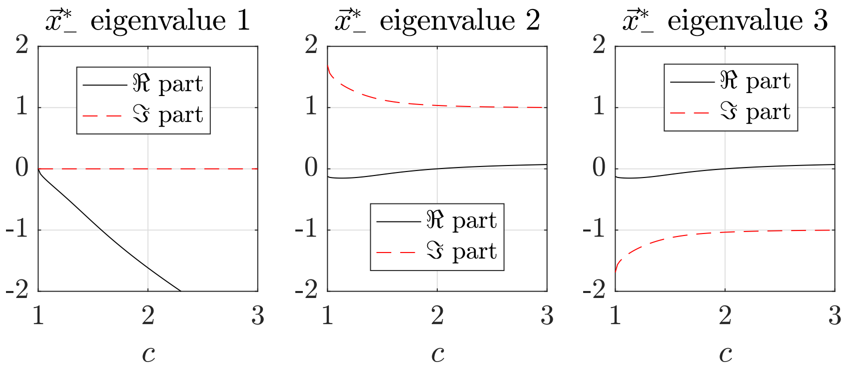
See the code in file `check_linearisation.m`. The orthonormal basis can be found analytically using the Gram-Schmidt process. Numerically one can use the Matlab command `orth`, which returns a matrix whose column vectors are an orthonormal basis of the space spanned by the column vectors of the input matrix.



The plots show that the real and imaginary part of the eigenvectors span the plane in which the spiralling occurs. There is indeed an exponentially decreasing spiral, linked to the complex conjugate eigenvalues with a negative real part. In the normal direction, the system initially moves a distance of the order of 10^{-3}, then seems to stabilise. The initial fast motion is a non-linear effect, which becomes less important near the fixed point. Making the simulation longer shows that the system will eventually converge to the fixed point or diverge to infinity. This is because the dynamics are slower (not exponential) in the normal direction.

(h) Still using $a = 1/4$ and $b = 1$, we are interested in the eigenvalues of the linearised system about the fixed points as c is varied. On Matlab, start from $c = c_{\text{lim}}$. Then, increase c. At each value of c compute the coordinates of the fixed points and find the eigenvalues of the Jacobian matrix at the fixed point. Plot the eigenvalues as c is varied.
Help : At each iteration of c, the `roots` command may not give the eigenvalues in the same order, meaning you may need to sort them (possibly by comparing their imaginary parts).





See the code in `eigenvalue_plot.m`. The code makes an animation of the eigenvalues in the complex plane. There are also regular plots with real and imaginary parts as a function of c. At c_{lim} there is only one fixed point which continuously separates into two fixed points as c increases, explaining why the eigenvalues of $\vec{x}^*_\pm$ start identical but continuously separate.

(i) Observe the dependence on c of the real eigenvalues of the Jacobian matrix at the fixed points, near $c = c_{\text{lim}}$. Does this confirm the bifurcation you had predicted in (f) ?

At $c = c_{\text{lim}}$ a fixed point appears and then separates into two fixed points. Near the bifurcation both present a stable spiral resulting from the pair of complex conjugate eigenvalues. The other eigenvalue is real and is equal to zero. For the fixed point $\vec{x}^*_+$, as c increases, the real eigenvalue becomes positive, i.e. it displays an unstable manifold in the associated direction. For $\vec{x}^*_-$ the real eigenvalue becomes negative, presenting a stable manifold. This confirms that this is a saddle node bifurcation.

(j) Observe the complex conjugate eigenvalues of one of the fixed points, what type of bifurcation can we expect at $c = 2$?

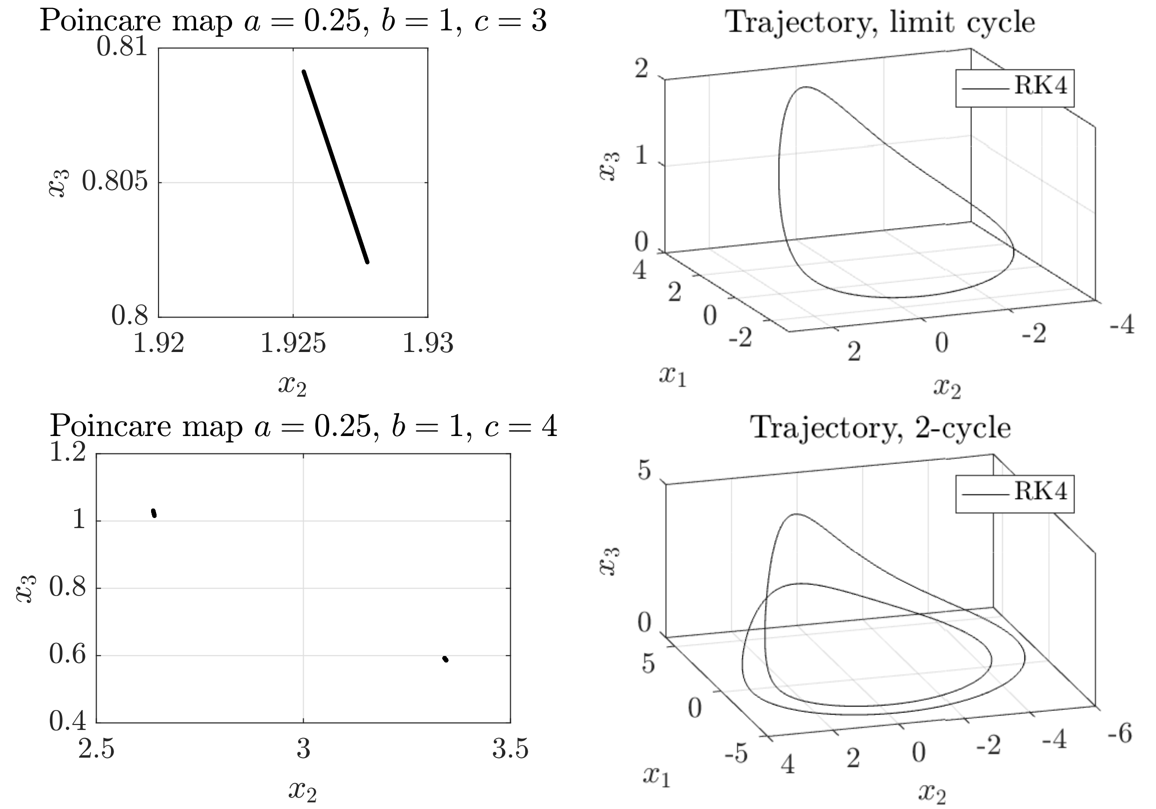
At $c = 2$, the real part of the complex conjugate pair of eigenvalues of $\vec{x}^*_-$ from negative becomes positive. As a consequence, the stable spiral becomes unstable. This is typical of a Hopf bifurcation.

(k) Discuss stability properties of the fixed points as $c \to \infty$.

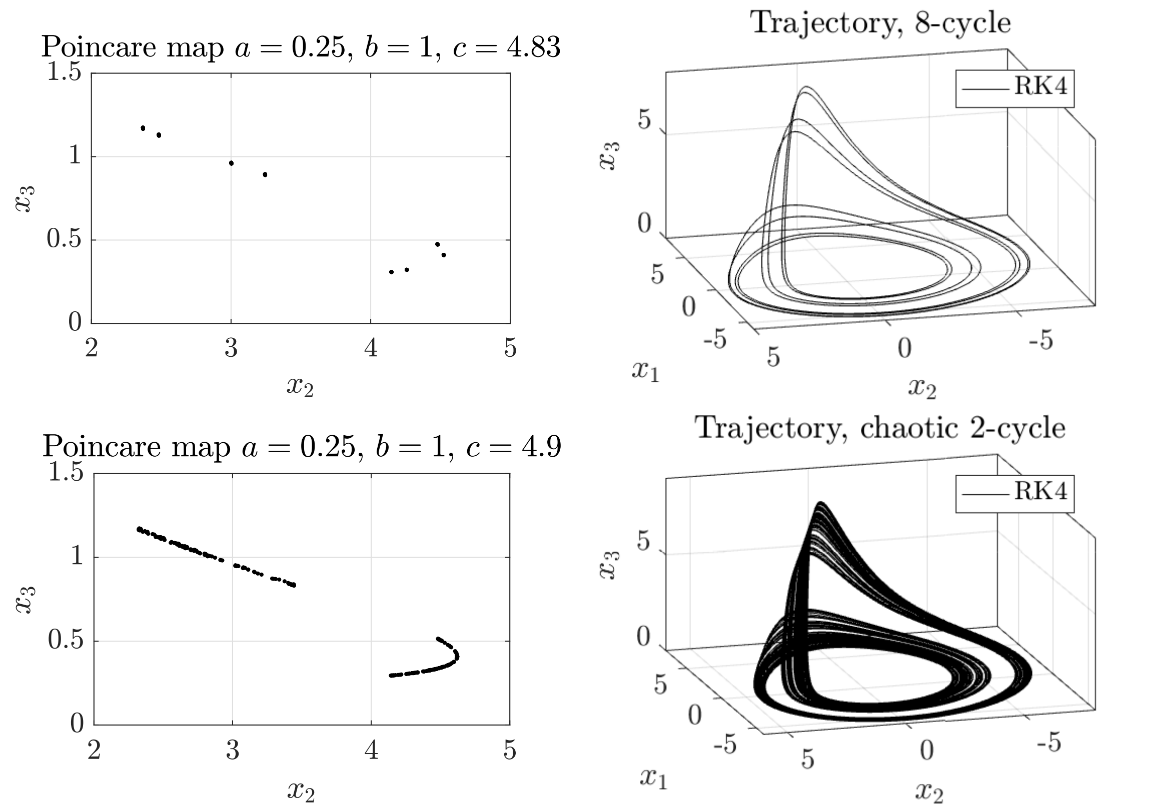
For $\vec{x}^*_-$ the real negative eigenvalue diverges to $-\infty$. Therefore the direction of the associated eigenvector becomes very stable. The complex conjugate pair converges to approximately $0.125 \pm 0.992\text{i}$. The associated plane presents an unstable spiral since the real part is positive. For $\vec{x}^*_+$ the complex conjugate pair diverges to $\pm\text{i}\infty$ and the associated plane trajectories are circles. The last eigenvalue converges to $1/4$, with an unstable associated manifold.

(l) Again with $a = 1/4$ and $b = 1$, using $c \in \{3, 4, 4.83, 4.9\}$ consider a few trajectories, and determine the long-term behaviour of the system (convergence to a stable fixed point, limit cycle, strange attractor, etc...). Then, plot the Poincaré map of the system in its final state. The Poincaré map is a 2D plot of the intersection between the trajectory and a given plane. We suggest to choose the $x_1 = 0$ plane.

See the code in file `poincare.m`. To plot the interesting parts of the Poincaré maps, one needs to first run a simulation, see the long time behaviour of the trajectories, in this case a limit cycle or a strange attractor, and launch a new simulation close to the final state. There is a simple limit cycle for $c = 3$. The Poincaré map is almost point-like, the small defects are due to the discretisation and numerical errors. For $c = 4$ there is also a stable orbit, which circles twice around the origin, something impossible in two dimensional systems as the trajectory cannot cross itself. Again the Poincaré map is ordered, this time the trajectories concentrate near two points, because the stable orbit circles twice around the origin.



For $c = 4.83$ the system is has a stable orbit, which this time circles eight times around the origin. The Poincaré map shows 8 points.



At the value of $c = 4.9$ the system is chaotic. The strange attractor towards which the system converges makes up a ruban which similarly to $c = 4$ circles twice around the origin.

It is actually a Möbius band as the surface twists on itself and cannot be oriented. The Poincaré map shows the two parts of the ruban. From our knowledge of strange attractors, inside the bands there should be a fractal pattern of infinite lines. This plot is limited by the numerical precision that we set, making it difficult to see the patterns. The increase of the number of times the limit cycle circles around the origin recalls the bifurcation process that we will see in the one-dimensional maps.